Fair resource allocation problems: A study of the nested cost sharing problem

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Research agenda

Resource Allocation Problems

- Toys resolution problem
- School choice problem
- Organ transplant problem
- Matching problem
- Queueing problem
- Dividend (Profits) allocation problem
- Estate division problem
- Bankruptcy problem (originated from two puzzles in Jewish document, the so-called "Talmud")

Research agenda

Contest Garment Problem

Worth of the garment	Claimant 1	Claimant 2
	100	200
200	50	150

Research agenda

Estate Division Problem

Estate of the man	Wife 1	Wife 2	Wife 3
	100	200	300
100	$\frac{100}{3}$	$\frac{100}{3}$	$\frac{100}{3}$
200	50	75	75
300	50	100	150

Research agenda

- King Solomon's problem
- Cake division (Pie-cutting) problem
- Nuclear power plant (or refuse burner) location problem
- Metro station location problem
- Social choice problem
- Power distribution (apportionment) problem
- Gerrymandering problem (Redistricting problem)
- *etc.*

Research agenda

- Each solution (rule) represents a core value system.
- Logical relations between core value and "fairness" criteria.
- A core value can be represented or equivalent to the implications of various combinations of fairness criteria.
- What makes one solution (core value) different from others.

Research approaches

1. Definition: simplicity and intuition.
2. Axiomatic approach: The departure point of the approach is the fairness properties. These properties are formally used to compare solutions. The ultimate object of the axiomatic study is to understand the implications of various combinations of different fairness properties. It is a centralized system.

Research approaches

3. Strategic approach: The socially desirable outcome recommended by the solution (or the core value) that is justified by axiomatic approach can be achieved through designing a non-cooperative game in which agents behave based on their own interests. It is a decentralized system. The axiomatic and strategic approaches are complement to each other. The departure point of the approach is to bridge the gap between the two counterparts (namely, cooperative and non-cooperative) of game theory.

Research approaches

In this project, we adopt axiomatic and strategic approaches to investigate a class of nested cost sharing problems, which has many real-life applications. This class of cost sharing problems has been studying for many years. One famous example is the study of irrigation ditches located in south-central Montana, USA.

Motivation and applications

Irrigation Ditch Problem

- Ranchers are distributed along an irrigation ditch.
- The rancher closest to the headgate only needs that the segment from the headgate to his field, the "first segment", be maintained, the second closest rancher needs that the first two segments be maintained, and so on.
- The cost of maintaining a segment used by several agents is incurred only once, independently of how many agents use it.

How should the maintenance cost of the ditch be shared among the ranchers?

Motivation and applications

Taxi-Fare (Uber) Sharing Problem

- Several agents are jointly riding a taxi. Different agents have different destinations.
- The further the destination an agent has, the longer the distance the agent needs.
- The taxi that accommodates a given agent with a certain distance accommodates any shorter distance that any agent has at no extra cost.

How should the tax-fare be shared among them?

Motivation and applications

Airport Runway Problem

- Several airlines are jointly using a runway. Different airlines need runways of different lengths.
- The larger the planes an airline operates, the longer the runway it needs.
- A runway accommodating a given plane accommodates any smaller plane any airline operates at no extra cost.
- A runway should be built long enough for the use of the largest plane.

How should the maintenance cost of the airstrip be shared among the airlines?

Other applications

- Elevator maintenance cost sharing problem
- Highway user fee problem
- Public transportation ticket pricing problem (including bus, train, subway,...)
- Electricity (Water) bill problem
- *etc.*

Motivation and applications

General applications

- Users in a group are linearly ordered by their needs for a facility (such as irrigation ditch, taxi, elevator, highway, bus, and *ect*.).
- Accommodating a user implies accommodating all users who "come before him" at no extra cost.
- The facility should satisfy a user with the largest need.

How should the cost of building up or maintaining such facility be shared among the users?

The model: formal definition

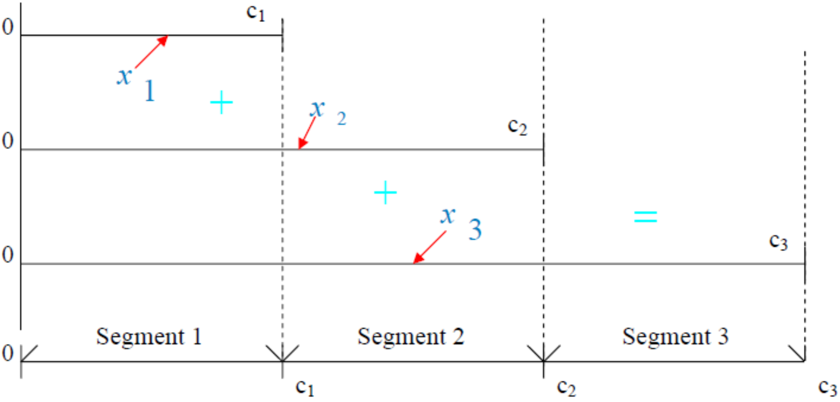
As mentioned, this is not brand-new problem. However, there was no formal discussion and rigorous analysis about this class of cost sharing problems until Littlechild and Owen (1973).

$$\varphi\left(N \equiv \{1,2,3\}, c \equiv (c_1, c_2, c_3) \in \mathbb{R}^N_+\right)$$

$$= (x_1, x_2, x_3) \in \mathbb{R}^N \text{s.t. for each } i \in N,\ 0 \le x_i \le c_i \text{ and } \sum_{i \in N} x_i = \max_{j \in N} c_j.$$

- The property, $0 \le x_i \le c_i$, is referred to as reasonableness and says that agent i should not receive a subsidy and should not contribute more than his cost parameter (the stand-alone cost).
- The property, $\sum_{i \in N} x_i = \max_{j \in N} c_j$, is referred to as efficiency and says that a rule should collect the exact amount of money to complete the work.

The model: geometric representation



$$(x_1, x_2, x_3) = \varphi\left(\{1,2,3\}; c_1, c_2, c_3\right)$$

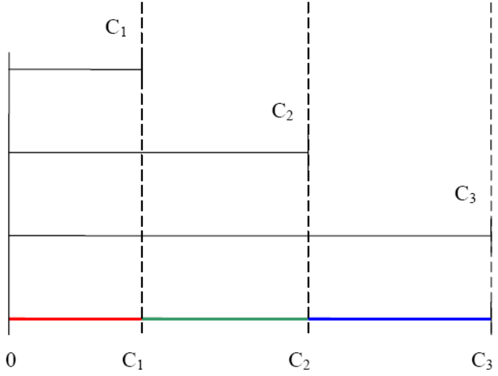
The Shapley value

Marginalism principle

The first rule says the following:

- All agents using a given segment contribute equally to its segmental cost.
- Each agent's contribution is the sum of terms, one term for each of the segments he uses.

The Shapley value



Agent 1: $\frac{c_1}{3} + 0 + 0$

Agent 2: $\frac{c_1}{3} + \frac{c_2 - c_1}{2} + 0$

Agent 3: $\frac{c_1}{3} + \frac{c_2 - c_1}{2} + c_3 - c_2$

The constrained equal benefits rule

When handing this class of the cost sharing problems, two aspects can be taken: how much agents have to contribute and how much agents can benefit from joining the cost sharing project.

The Shapley value focuses on the former.

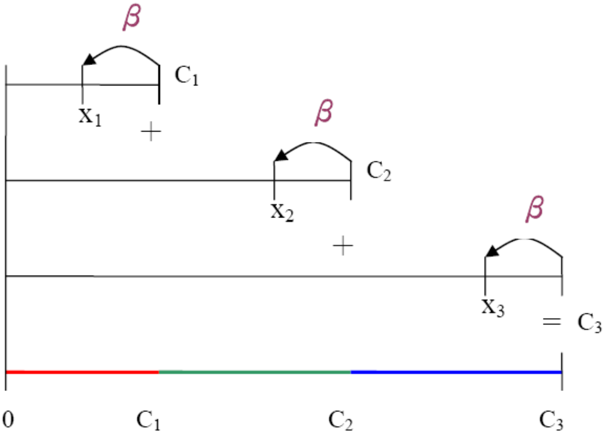
The next rule (the constrained equal benefits rule) focuses on the latter.

The constrained equal benefits rule

Egalitarianism principle on benefit

It equalizes agents' benefits subject to no one contributing a negative amount.

The constrained equal benefits rule



The egalitarian solution

Egalitarianism principle on contribution

The rule can be understood as follows. Start by requiring that all agents in N should contribute equally until there are $\lambda^1 \in \mathbb{R}_+$ and a group $\{1, \cdots, l^1\}$ such that $\lambda^1 l^1 = c_{l^1}$. Each agent in $\{1, \cdots, l^1\}$ then contributes λ^1. The algorithm next requires that all agents in $\{l^1 + 1, \cdots, n\}$ should contribute equally until there are $\lambda^2 \in \mathbb{R}_+$ and a group $\{l^1 + 1, \cdots, l^2\}$ such that $\lambda^2 \left(l^2 - l^1\right) = c_{l^2} - c_{l^1}$. Each agent in $\{l^1, \cdots, l^2\}$ then contributes λ^2. Continue this process until the total cost c_n is covered.

The egalitarian solution

Thanks to the linear structure of the nested cost sharing problems, the allocation chosen by the egalitarian solution can be obtained by the following formula (Aadland and Kolpin, 1998).

Egalitarian solution, $\boldsymbol{E}$: For each $(N, c) \in \mathcal{A}$,

$$
\begin{aligned}
E_1(N,c) \equiv & \ \min_{1 \leq k \leq n} \left\{ \frac{c_k}{k} \right\} \\
E_i(N,c) \equiv & \ \min_{i \leq k \leq n} \left\{ \frac{c_k - \sum_{p=1}^{i-1} E_p(N,c)}{k-i+1} \right\} \quad \text{where } 2 \leq i \leq n-1 \\
E_n(N,c) \equiv & \ c_n - \sum_{p=1}^{n-1} E_p(N,c) .
\end{aligned}
$$

The nucleolus

John Rawls (or maximin) principle

The next rule lexicographically maximizes the "welfare" of the worst-off coalitions. Since for general games, the payoff vector it chooses is obtained by solving a sequence of linear programs, it is in general not easy to compute. However, for the nested cost sharing problems, the allocation chosen by the nucleolus can be calculated by an explicit formula (Sönmez, 1994).

Nucleolus, ***Nu***: For each $(N, c) \in \mathcal{A}$,

$$\begin{aligned} Nu_1(N,c) &\equiv \min_{1\leq k\leq n-1}\left\{\frac{c_k}{k+1}\right\} \\ Nu_i(N,c) &\equiv \min_{i\leq k\leq n-1}\left\{\frac{c_k-\sum_{p=1}^{i-1}Nu_p(N,c)}{k-i+2}\right\}, \quad \text{where } 2\leq i\leq n-1 \\ Nu_n(N,c) &\equiv c_n-\sum_{p=1}^{n-1}Nu_p(N,c). \end{aligned}$$

Equal treatment of equals

Equal treatment of equals is a simple and basic fairness requirement. It says that agents with the same cost parameters should contribute equal amounts.

Last-agent cost additivity

Last-agent cost additivity has to do with possible increase of the cost parameter of the last agent. It says that if the cost parameter of the last agent increases by δ, then his contribution should increase by δ.

Cost monotonicity

Cost monotonicity says that if an agent's cost parameter increases, then all other agents should contribute at most as much as they did initially. In contrast to pollution emission problems, when an agent's cost parameter increases, this change will lead to an increase in other agent's cost and the total cost, and be considered as a negative externality. However, in our model, this change will not lead to an increase in other agent's cost and the total cost, except for the last agent's cost parameter and be considered as a positive externality. This is because the cost structure is nested and the facility is a kind of local public good or club good. In addition, when an agent's cost parameter increases, it means this agent will demand the facility more and should contribute more, and all other agents will reduce their burdens.

Axiomatic justifications for the Shapley value

Theorem: (IJET, 2012) $\varphi:$ efficiency$+$ equal share lower bound$+$ cost monotonicity$+$ population fairness $\Leftrightarrow \varphi = Sh$.

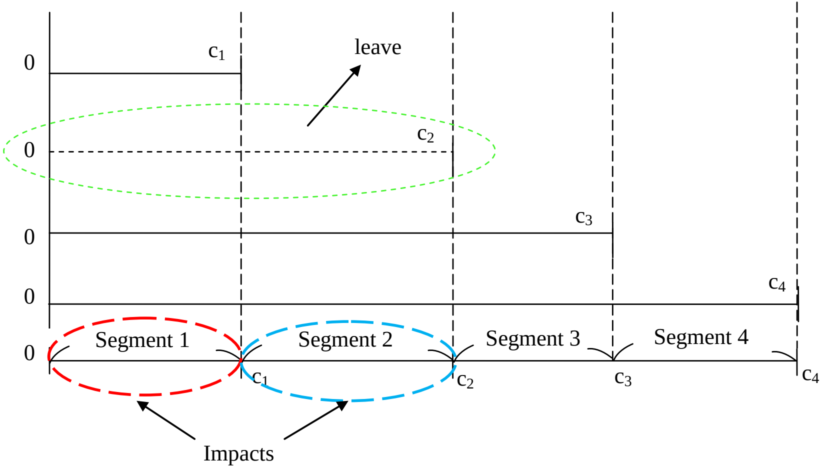
Theorem: (IJET, 2012) $\varphi:$ efficiency (reasonableness)$+$ equal share lower bound$+$ cost monotonicity$+$ smallest-cost consistency $\Leftrightarrow \varphi = Sh$.

Theorem: (IJET, 2012) $\varphi:$ efficiency$+$ balanced population impact $\Leftrightarrow \varphi = Sh$.

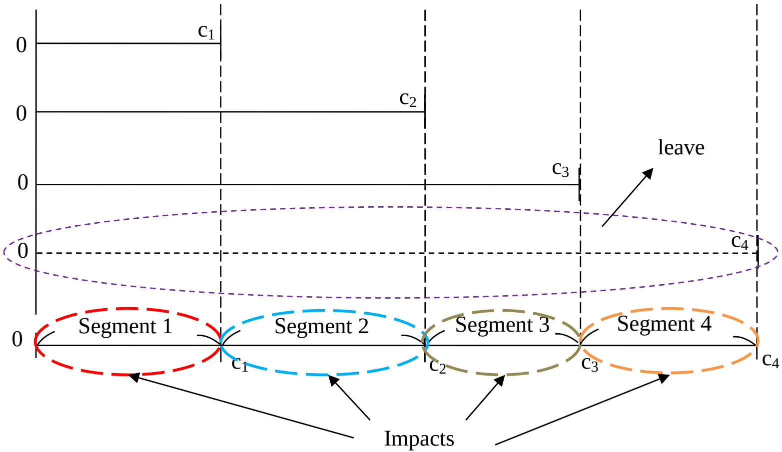
Strategic justifications for the Shapley value

As suggested by Krishna and Serrano (1996), the properties of a solution can be used as guides in designing a non-cooperative game that implements the solution. We follow this tradition and use properties of the rule to design a non-cooperative game that gives a strategic justification for the Shapley value. In particular, the property balanced population impact will play a central role in this regard. It says that if an agent leaves an airport problem, this will typically affect the contributions of other remaining agents. The balanced population impact property requires that the effect of agent i leaving on the contribution of another agent $j \neq i$ should be equal to the effect of agent j leaving on the contribution of agent i.

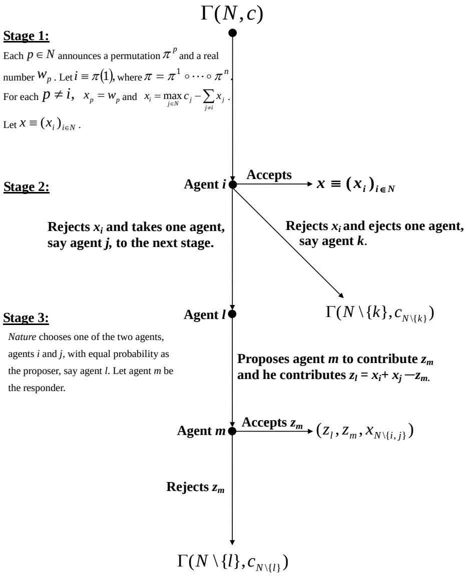
Strategic justification for the Shapley value



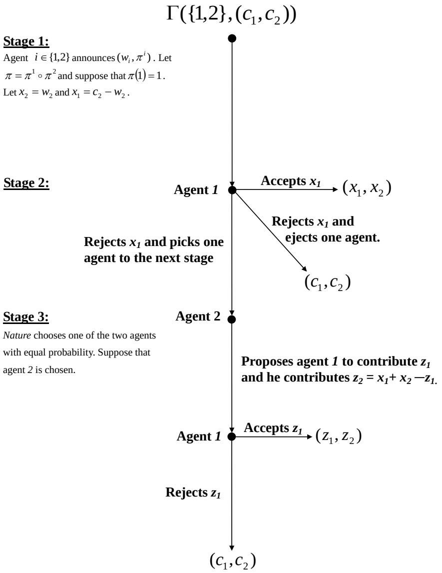
Strategic justification for the Shapley value



Strategic justifications for the Shapley value



Strategic justification for the Shapley value



Strategic justification for the Shapley value

$$\begin{array}{ccc} N & & \pi^1 \\ \hline 1^* & \longrightarrow & 1 \\ & & \\ 2 & \longrightarrow & 2 \end{array}$$

and

$$\begin{array}{ccc} N & & \pi^2 \\ \hline 1^* & \longrightarrow & 1 \\ & & \\ 2 & \longrightarrow & 2 \end{array}$$

Strategic justification for the Shapley value

$$
\begin{array}{cccccc}
N & & \pi^2 & \circ & \pi^1 \equiv \pi & \\
\hline
1^* & \longrightarrow & 1 & \longrightarrow & 1 & \pi(1^*) \\
 & & & & & \\
2 & \longrightarrow & 2 & \longrightarrow & 2 & \pi(2)
\end{array}
$$

Strategic justification for the Shapley value

$$\begin{tikzcd}
\underline{N} & \underline{\pi^1} \\
1^* \arrow[rd] & 1 \\
2 \arrow[ru] & 2
\end{tikzcd}$$

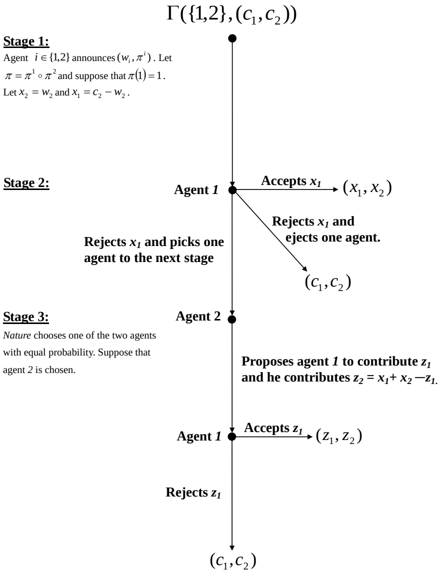
and

$$\begin{tikzcd}
\underline{N} & \underline{\pi^2} \\
1^* \arrow[r] & 1 \\
2 \arrow[r] & 2
\end{tikzcd}$$

Strategic justification for the Shapley value

$$\begin{tikzcd}
N & \pi^2 \quad \circ & \pi^1 \equiv \pi & \\
1^* \arrow[r, red] & 1 \arrow[dr, red] & 1 & \pi(2) \\
2 \arrow[r] & 2 \arrow[ur] & 2 & \pi(1^*)
\end{tikzcd}$$

Strategic justification for the Shapley value



Strategic justification for the Shapley value

The idea of designing Stage 3 comes from the balanced population impact property. As shown in Chun et al. (2012), the Shapley value satisfies this property. Thus,

$$
\begin{aligned}
& Sh_1(\{1,2\},(c_1,c_2)) - Sh_1(\{1,2\}\setminus\{2\},c_1) \\
= {} & Sh_2(\{1,2\},(c_1,c_2)) - Sh_2(\{1,2\}\setminus\{1\},c_2).
\end{aligned}
$$

It can be rewritten as

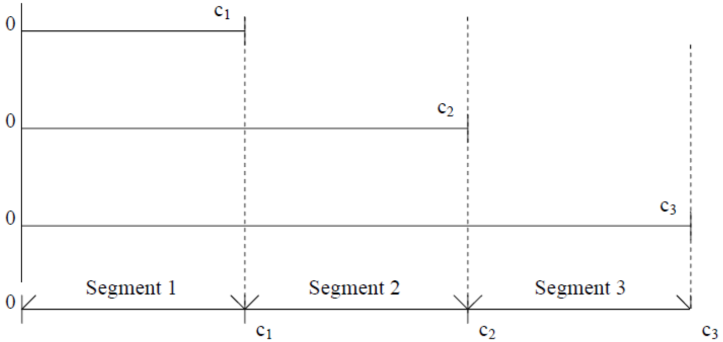
$$
\begin{aligned}
& Sh_1(\{1,2\},(c_1,c_2)) \\
= {} & \frac{1}{2}\Bigg\{\underbrace{Sh_1(\{1,2\},(c_1,c_2)) + Sh_2(\{1,2\},(c_1,c_2)) - Sh_2(\{1,2\}\setminus\{1\},c_2)}_{\text{When agent 1 is the proposer, agent 1's contribution.}} \\
& + \underbrace{Sh_1(\{1,2\}\setminus\{2\},c_1)}_{\text{When agent 2 is the proposer, agent 1's contribution.}}\Bigg\}.
\end{aligned}
$$

Strategic justifications for the Shapley value

Theorem: (Existence result) There is a subgame perfect equilibrium of $\Gamma_{Sh}(N, c)$ with outcome $Sh(N, c)$.

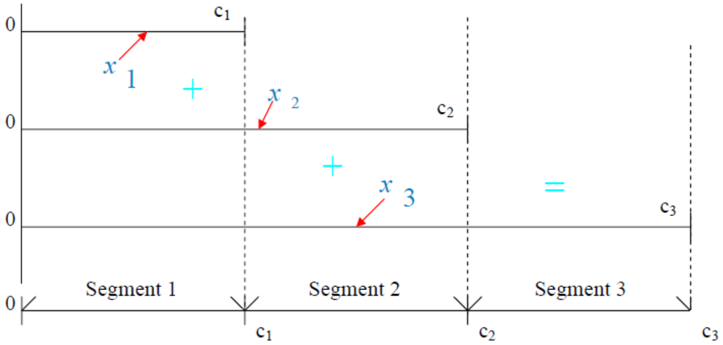
Theorem: (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_{Sh}(N, c)$ is $Sh(N, c)$.

Bilateral consistency and converse consistency



There are three airlines. For each airline, say airline *i*, the cost of satisfying its need is represented by the cost parameter c_i. For simplicity, assume that $c_1 < c_2 < c_3$. Thus, airline 1 uses segment 1. Airline 2 uses segments 1 and 2. Airline 3 uses segments 1, 2 and 3. The cost of segment 1 is c_1, the cost of segment 2 is $c_2 - c_1$, the cost of segment 3 is $c_3 - c_2$. The total cost of the airstrip is c_3, which is the sum of c_1 (the cost of segment 1), $c_2 - c_1$ (the cost of segment 2), and $c_3 - c_2$ (the cost of segment 3).

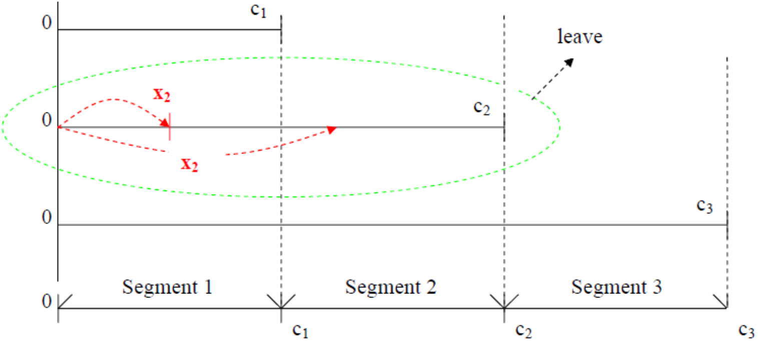
Bilateral consistency and converse consistency



$$(x_1, x_2, x_3) = \varphi\left(\{1,2,3\}; c_1, c_2, c_3\right)$$

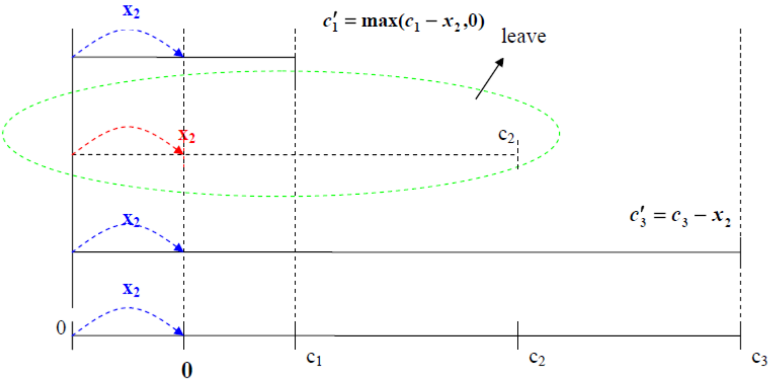
Bilateral consistency and converse consistency

LS formulation:



Bilateral consistency and converse consistency

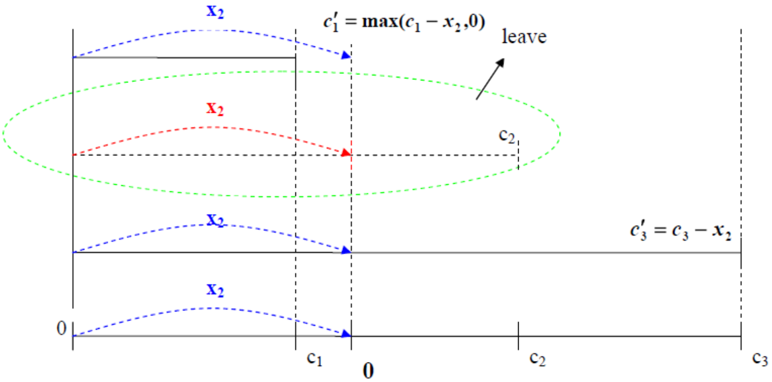
LS bilateral consistency:



$$x_1 = \varphi_1(\{1,3\}; c_1', c_3') \text{ and } x_3 = \varphi_3(\{1,3\}; c_1', c_3')$$

Bilateral consistency and converse consistency

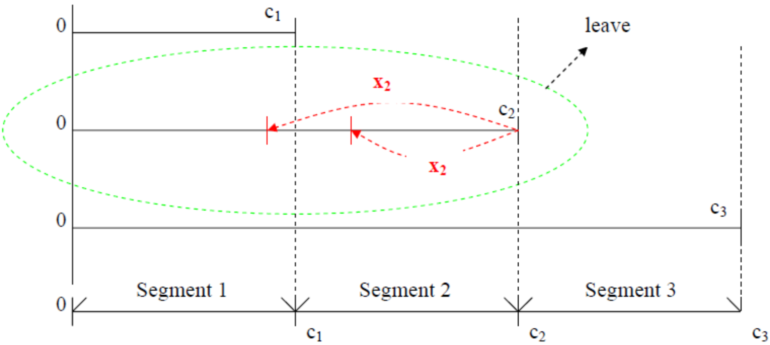
LS bilateral consistency:



$$x_1 = \varphi_1(\{1,3\}; c_1', c_3') \text{ and } x_3 = \varphi_3(\{1,3\}; c_1', c_3')$$

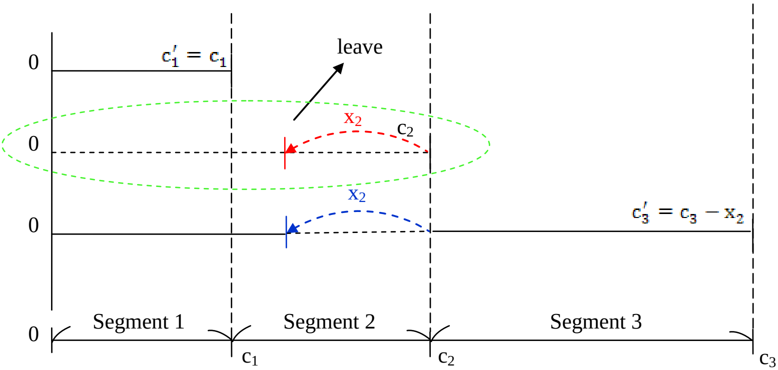
Bilateral consistency and converse consistency

RS formulation:



Bilateral consistency and converse consistency

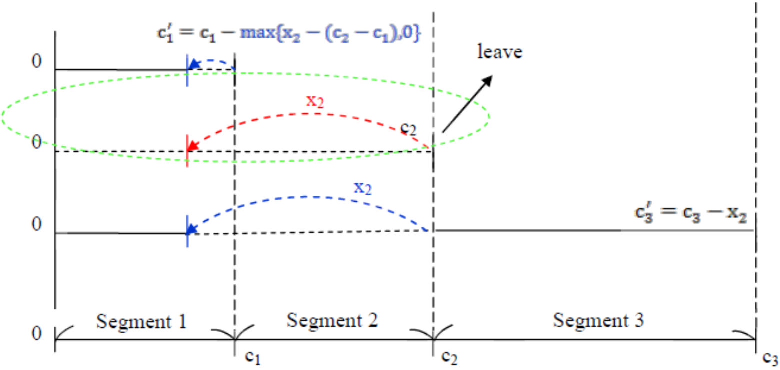
RS bilateral consistency:



$$x_1 = \varphi_1\left(\{1,3\}; c'_1, c'_3\right) \text{ and } x_3 = \varphi_3\left(\{1,3\}; c'_1, c'_3\right)$$

Bilateral consistency and converse consistency

RS bilateral consistency:



$x_1 = \varphi_1\left(\{1,3\}; c'_1, c'_3\right)$ *and* $x_3 = \varphi_3\left(\{1,3\}; c'_1, c'_3\right)$

Bilateral consistency and converse consistency

- We adopt LS formulation to propose the **reduced problem of (N, c) with respect to $N' \equiv \{i, n\}$ and x**, $(N', r^x_{N'})$, is defined by setting

$$(r^x_{N'})_i \equiv \max\left\{c_i - \textstyle\sum_{k \neq i,n} x_k, 0\right\} \text{ and}$$
$$(r^x_{N'})_n \equiv c_n - \textstyle\sum_{k \neq i,n} x_k.$$

  - **LS bilateral consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| \geq 2$ and each $i \in N \setminus \{n\}$, if $x = \varphi(N, c)$, then $(\{i, n\}, r^x_{\{i,n\}}) \in \mathcal{A}$ and $x_{\{i,n\}} = \varphi(\{i, n\}, r^x_{\{i,n\}})$.
  - **LS converse consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| > 2$ and each $x \in X(N, c)$, if for each $N' \subset N$ with $|N'| = 2$ and $n \in N'$, $x_{N'} = \varphi(N', r^x_{N'})$, then $x = \varphi(N, c)$.

Bilateral consistency and converse consistency

- We also adopt RS formulation to propose the **reduced problem of (N, c) with respect to $N' \equiv \{i, n\}$ and x**, $(N', r^x_{N'})$, is defined by setting

$$(r^x_{N'})_i \equiv \max\left\{\min_{i\leq k, k\neq n}\left\{c_k - \sum_{m\leq k, m\neq i,n} x_m\right\}, 0\right\} \text{ and}$$

$$(r^x_{N'})_n \equiv \max\left\{\min_{n\leq k, k\neq i}\left\{c_k - \sum_{m\leq k, m\neq i,n} x_m\right\}, 0\right\} = c_n - \sum_{m\neq i,n} x_m.$$

Bilateral consistency and converse consistency

- **RS bilateral consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| \geq 2$ and each $i \in N \setminus \{n\}$, if $x = \varphi(N, c)$, then $(\{i, n\}, r^x_{\{i,n\}}) \in \mathcal{A}$ and $x_{\{i,n\}} = \varphi(\{i, n\}, r^x_{\{i,n\}})$.
- **RS converse consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| > 2$ and each $x \in X(N, c)$, if for each $N' \subset N$ with $|N'| = 2$ and $n \in N'$, $x_{N'} = \varphi(N', r^x_{N'})$, then $x = \varphi(N, c)$.

Bilateral consistency and converse consistency

when we restrict attention to the departure of the first agent, RS consistency and LS consistency coincide.

First-agent consistency: For each $(N, c) \in \mathcal{A}$ with $|N| \geq 2$ and each $i \in N \setminus \{1\}$, if $x = \varphi(N, c)$, then $(N \setminus \{1\}, r^x_{N \setminus \{1\}}) \in \mathcal{A}$ and $x_{N \setminus \{1\}} = \varphi(N \setminus \{1\}, r^x_{N \setminus \{1\}})$.

Axiomatic justifications for the CEB rule

We now turn to the CEB rule.

Theorem: φ : order preservation for benefits+ cost monotonicity+ LS bilateral consistency $\Leftrightarrow \varphi = CEB$.

Theorem: φ : order preservation for benefits+ cost monotonicity+ LS converse consistency $\Leftrightarrow \varphi = CEB$.

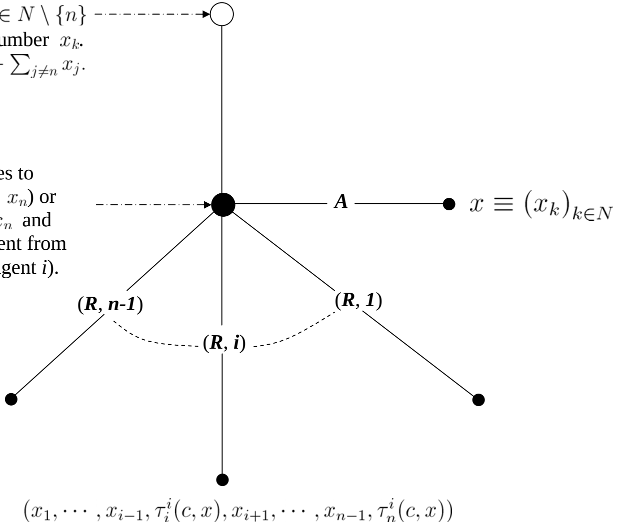
Theorem: φ : equal treatment of equals+ last-agent cost additivity+ LS bilateral consistency $\Leftrightarrow \varphi = CEB$.

Theorem: φ : equal treatment of equals+ last-agent cost additivity+ LS converse consistency $\Leftrightarrow \varphi = CEB$.

Strategic justifications for the CEB rule

Stage 1:
Each agent $k \in N \setminus \{n\}$ announces a number x_k.
Let $x_n \equiv c_n - \sum_{j \neq n} x_j$.

Stage 2:
Agent *n* decides to take ***A*** (accept x_n) or (***R***, ***i***) (reject x_n and choose one agent from $N \setminus \{n\}$, say agent *i*).



Strategic justification for the CEB rule

The contributions of agent n and agent i are specified as follows. Imagine that there is a fair coin to select one of the two agents. The chosen one, say agent $l \in \{i, n\}$, has no choice but to pick the group $N \backslash \{i, n\}$ and takes the sum of their contributions $\sum_{k \in N \backslash \{i,n\}} x_k$ to cover her cost c_l (the cost of building the part of the runway agent l can use). Namely, she contributes $\max \left\{0, c_l - \sum_{k \in N \backslash \{i,n\}} x_k \right\}$. The other agent contributes $x_i + x_n - \max \left\{0, c_l - \sum_{k \in N \backslash \{i,n\}} x_k \right\}$.

Strategic justifications for the CEB rule

Theorem: (GEB, 2012) (Existence result) There exists a subgame perfect equilibrium of $\Gamma(N, c)$ with outcome $CEB(N, c)$.

Theorem: (GEB, 2012) (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma(N, c)$ is $CEB(N, c)$.

Axiomatic justifications for the nucleolus

We next turn to the nucleolus and show the following:

Theorem: φ: equal treatments of equals + last-agent cost additivity + RS bilateral consistency (or RS converse consistency) $\Leftrightarrow \varphi = Nu$.

Compared to our previous results:

Theorem: φ: equal treatments of equals + last-agent cost additivity + LS bilateral consistency (or LS converse consistency) $\Leftrightarrow \varphi = CEB$.

Axiomatic justification for the nucleolus

The characterizations of the CEB rule and the nucleolus pin down the essential differences between the two rules from axiomatic viewpoint. They are LS (RS) bilateral consistency and LS (RS) converse consistency.

Strategic justification for the nucleolus

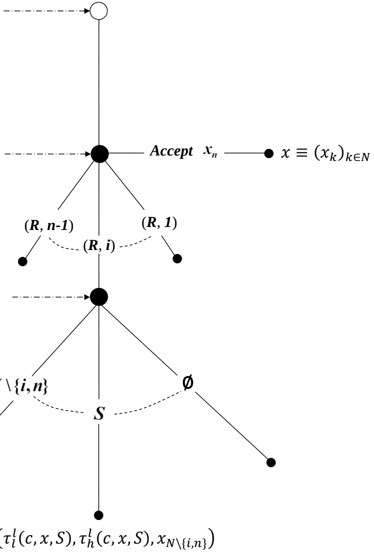
Inspired by the above axiomatic characterization results, we ask whether different bilateral consistency and converse consistency properties pin down the essential differences between the nucleolus and the CEB rule from strategic viewpoint.

Strategic justification for the nucleolus

<u>Stage 1:</u>
Each agent $k \in N \backslash \{n\}$ announces a number $x_k \in \mathbb{R}_+$.
Let $x_n = c_n - \sum_{k \neq n} x_k$.

<u>Stage 2:</u>
Agent n decides to ***accept*** $\boldsymbol{x_n}$ or take $(\boldsymbol{R, i})$ (reject x_n and choose one agent from $N \backslash \{n\}$, say agent i).

<u>Stage 3:</u>
A fair coin selects one agent from $\{i, n\}$, say agent l. Agent l picks a group of agents from $2^{N \backslash \{i,n\}}$, say $\boldsymbol{S}$.
Let $h \in \{i, n\} \backslash \{l\}$.



Strategic justification for the nucleolus

The contributions of agents i and n are specified as follows. Imagine that there is a fair coin to select one of the two agents. The chosen one, say agent $l \in \{i, n\}$, picks a group, say S, from $2^{N \setminus \{i,n\}}$ and takes $\sum_{k \in S} x_k$ to cover $\max_{k \in S \bigcup \{l\}} \{c_k\}$ (the cost of building the part of the runway agent l and all agents in S can use). Namely, her contribution is $\max \left\{ \max_{k \in S \bigcup \{l\}} \{c_k\} - \sum_{k \in S} x_k, 0 \right\}$. The other agent contributes the remainder

$x_i + x_n - \max \left\{ \max_{k \in S \bigcup \{l\}} \{c_k\} - \sum_{k \in S} x_k, 0 \right\}$.

Strategic justification for the nucleolus

In the CEB game, agent l has no choice but to pick the group $N \backslash \{i, n\}$ and takes the sum of their contributions $\sum_{k \in N \backslash \{i,n\}} x_k$ to cover her cost c_l (the cost of building the part of the runway agent l can use). Namely, she contributes $\max \left\{0, c_l - \sum_{k \in N \backslash \{i,n\}} x_k \right\}$. The other agent contributes the remainder $x_i + x_n - \max \left\{0, c_l - \sum_{k \neq i,n} x_k \right\}$. In the nucleolus game, agent l picks the group S from $2^{N \backslash \{i,n\}}$ and takes the sum of their contributions $\sum_{k \in S} x_k$ to cover $\max_{k \in S \bigcup \{l\}} \{c_k\}$ (the cost of building the part of the runway agent l and all agents in S can use). Namely, her contribution is $\max \left\{0, \max_{k \in S \bigcup \{l\}} \{c_k\} - \sum_{k \in S} x_k \right\}$. The other agent contributes the remainder $x_i + x_n - \max \left\{\max_{k \in S \bigcup \{l\}} \{c_k\} - \sum_{k \in S} x_k, 0 \right\}$.

Strategic justification for the nucleolus

Theorem: (Existence result) There exists a subgame perfect equilibrium of $\Gamma_{Nu}(N, c)$ with outcome $Nu\,(N, c)$.

Theorem: (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_{Nu}(N, c)$ is $Nu\,(N, c)$.

Strategic justification for the egalitarian solution

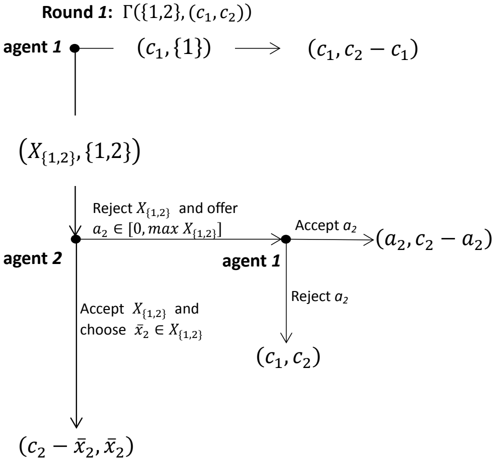
Recall the formula of the egalitarian solution. Start by requiring that all agents in N should contribute equally until there are $\lambda^1 \in \mathbb{R}_+$ and a group of agents $\{1, \cdots, l^1\}$ such that $\lambda^1 l^1 = c_{l^1}$. Each agent in $\{1, \cdots, l^1\}$ then contributes λ^1. The algorithm next requires that all agents in $\{l^1 + 1, \cdots, n\}$ should contribute equally until there are $\lambda^2 \in \mathbb{R}_+$ and a group of agents $\{l^1 + 1, \cdots, l^2\}$ such that $\lambda^2 (l^2 - l^1) = c_{l^2} - c_{l^1}$. Each agent in $\{l^1, \cdots, l^2\}$ then contributes λ^2. Continue this process until the total cost c_n is covered.

Strategic justification for the egalitarian solution

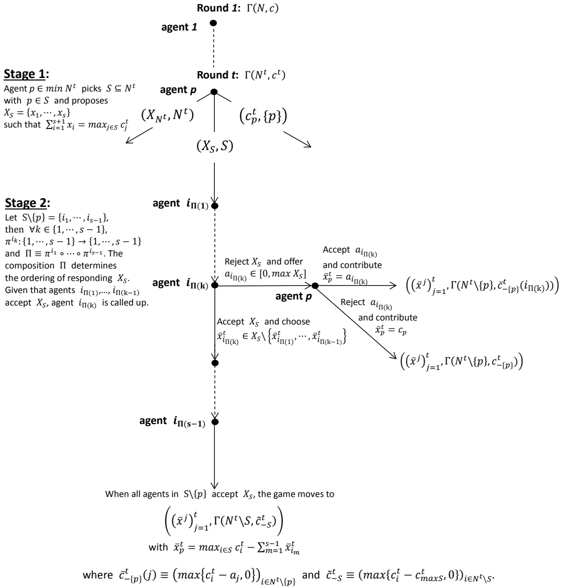
Stage 1:

Agent *1* picks $S \in \{\{1\},\{1,2\}\}$ and proposes c_1 if $S = \{1\}$, and $X_{\{1,2\}} = \{x_1, x_2\}$ such that $x_1 + x_2 = c_2$ if $S = \{1,2\}$.

Stage 2:



Strategic justification for the egalitarian solution



Strategic justification for the egalitarian solution

Theorem: (Existence result) There exists a subgame perfect equilibrium of $\Gamma_E(N, c)$ with outcome $E(N, c)$.

Theorem: (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_E(N, c)$ is $E(N, c)$.

The egalitarian solution vs the nucleolus

What makes the nucleolus different from the egalitarian solution. It is well-known that the nucleolus satisfies the property of last-agent cost additivity but the egalitarian solution does not. We exploit this difference to obtain a new game from the one implementing the egalitarian solution. How to do it? We propose to exclude the participation of the last agent in Stage 1 of each round and suggest that after collecting all other agents' contributions, agent n contributes the residual cost (the difference between the total cost and the total contribution already made).

The egalitarian solution vs the nucleolus

Inspired by the formulae of the nucleolus and the egalitarian solution, another difference between the two solutions is: the denominator of each term in the formula of the nucleolus is incremented by one, compared to the denominator of the corresponding term in the formula of the egalitarian solution. We revise the game that implements the egalitarian solution based on the two differences as follows.

The egalitarian solution vs the nucleolus

In Stage 1: The last agent is not a potential agent anymore in each round. Instead,

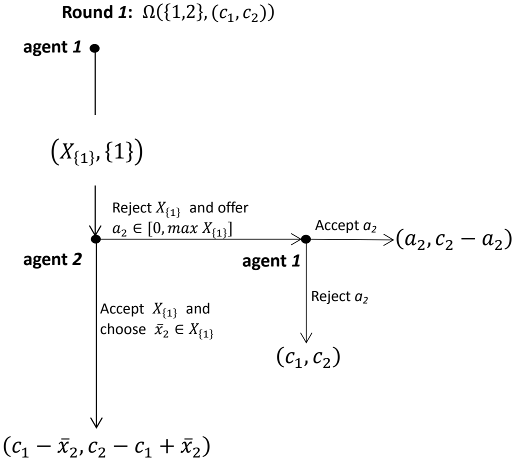
In Stage 2: The last agent plays in Stage 2 of each round as a role of helper to reduce the total contribution made by the group chosen by the first agent in each round. However, the last agent's contribution is determined after collecting all other agents‘ contributions. He then contributes the residual cost.

The egalitarian solution vs the nucleolus

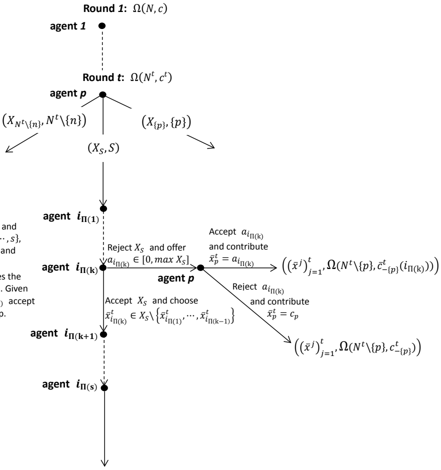
Stage 1:

Agent *1* picks $\{1\}$ and proposes $X_{\{1\}} = \{x_1, x_2\}$ such that $x_1 + x_2 = c_1$.

Stage 2:



Stage 1:
Agent $p \in min\ N^t$ picks $S \subseteq N^t \backslash \{n\}$ with $p \in S$, and proposes $X_S = \{x_1, \cdots, x_{s+1}\}$ such that $\sum_{i=1}^{s+1} x_i = max_{j \in S}\ c_j^t$.



Stage 2:
Let $S\backslash\{p\} = \{i_1, \cdots, i_{s-1}\}$, and let $i_s = n$, then $\forall k \in \{1, \cdots, s\}$, $\pi^{i_k}: \{1, \cdots, s\} \to \{1, \cdots, s\}$ and $\Pi \equiv \pi^{i_1} \circ \cdots \circ \pi^{i_s}$. The composition Π determines the ordering of responding X_S. Given that agents $i_{\Pi(1)}, \ldots, i_{\Pi(k-1)}$ accept X_S, agent $i_{\Pi(k)}$ is called up.

When all agents in $(S \cup \{n\})\backslash\{p\}$ accept X_S, the game moves to

$$\left((\bar{x}^j\backslash\{\bar{x}_n^j\})_{j=1}^t, \Omega(N^t\backslash S, \hat{c}_{-S}^t)\right)$$

with $\bar{x}_p^t = max_{i \in S}\ c_i^t - \sum_{m=1}^{s} \bar{x}_{i_m}^t$

where $\bar{c}_{-\{p\}}^t(j) \equiv \left(max\{c_i^t - a_j, 0\}\right)_{i \in N^t\backslash\{p\}}$ and $\hat{c}_{-S}^t \equiv (max\{c_i^t - \sum_{k \in S} \bar{x}_k^t, 0\})_{i \in N^t\backslash S}$.

The egalitarian solution vs the nucleolus

We show:

Theorem: (Existence result) There exists a subgame perfect equilibrium of $\Gamma_{Nu}(N, c)$ with outcome $Nu\,(N, c)$.

Theorem: (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_{Nu}(N, c)$ is $Nu\,(N, c)$.

The egalitarian solution vs the nucleolus

The implementation results of the egalitarian solution and the nucleolus show that assigning different roles to the last agent leads to implementing different solutions. The results point out the difference between the two solutions from strategic perspective and establish a strategic comparison between the solutions. This is the first paper in the non-cooperative implementation literature to observe such surprising phenomenon.

Thank you!!