# Exit in Vertical Relationships Under Uncertainty *

Piin-hueih Chiang†

Department of Economics

National Chengchi University

September 30, 2013

**Abstract**

This paper studies the spillover effect of exit in a vertical relationship. I extend the methodology of irreversible investment under uncertainty to consider exits in a vertical market structure. When the exogenous demand shock is low, one party of the supply chain wants to exit first and will thus lead to the exit of the remaining party. The firm which wants to exit later strategically acts to delay the exit of its counterparty and therefore prevents its own exit. When the state level drops below the unique equilibrium exit threshold, both firms will exit simultaneously. The expected delay in exit timing is derived.

## 1 Introduction

During the past decade, different sorts of exogenous shocks struck the world. Some were regional and some had global impacts. In a supply chain, there exist

*I am grateful to Svetlana Boyarchenko for her continued advice and encouragement. I am thankful to Frank Reidel, Maxwell Stinchcombe, Thomas Wiseman and Dale Stahl for their advice, and I benefited from discussion and comments of participants of the micro-theory workshop in the department of economics, University of Texas at Austin. All errors are mine.

†Comments welcome at phchiang@nccu.edu.tw

multiple vertical relationships. If an exogenous shock hits one side of the vertical relationship, the pain passes up and down the supply chain inevitably. The following are three examples that motivated this paper.

The first example is the big earthquake and tsunami that hit Japan in March 2011 and the devastating flood that struck Thailand in October 2011. Before the electronics and automotive industries could recover from the shock of the Japan earthquake, they were struck again by a shortage of components produced by suppliers in central Thailand. Component shortages have forced Honda to cut down production around the world and decreased its revenue until Honda's recovery in March 2012. The launch of Sony's high-end Nex-7 camera was delayed from November 2011 to February 2012. Since the hard disk production is highly concentrated in Thailand, sales of PC after the Thai flood were impacted by supply constraints. These two natural catastrophes may be just a regional shock, but the impact of these shocks is global wide as the pain passes down the supply chain.

The U.S. subprime mortgage crisis was one of the first indicators of the late-2000 financial crisis. The subprime mortgage crisis began by a rise in subprime mortgage delinquencies and foreclosures, and resulted in decline of securities backed by said mortgages. It resulted in the collapse of large financial institutions, the bailout of banks by governments, and downturns in stock markets around the world. In many areas, the housing market also suffered, resulting in numerous evictions, foreclosures and prolonged unemployment. Consumer wealth and economic activity declined and led to a severe global economic recession in 2008.

Another example of passing pain down the chain is the typically combative relationships between retailers and suppliers. In the retail market, supermarkets are constantly trying to drive better deals from their suppliers. To increase sales, the retailers oftentimes look to make savings or pass on cheaper prices to consumers. As the converters (companies that take ingredients and then produce ready meals, sandwiches, cakes) buy from the ingredient suppliers and farmers, the pain is felt right down the supply chain when the retailers tries to cut its costs. Or it could go in the opposite direction. When the prices of ingredients are increasing, there is pressure to push the pain down the supply chain and as we can see the prices of the final products in the supermarkets increase as well.

In this paper, I extend the methodology of irreversible investment under uncertainty to consider exit in a vertical market structure. More particularly, I examine a declining downstream market with the demand of the final good evolving according to a geometric Brownian motion at such low drift that it eventually becomes optimal for the firms to exit. The decision to exit is irreversible. In this vertical structure, there is one upstream and one downstream monopolists, and they are initially not integrated. If one side of the vertical relationships exits, the counterparty must exit as well. When one firm is considering to exit, how will its upstream or downstream counterparty react to this exit? The question to be analyzed is how the firm strategically delays the exit of its counterparty to prevent its own exit.

The methodological feature of the model is that it incorporates in the real-options framework the subgame perfect equilibrium concept. I find the equilibrium exit timing when both of the firms exit simultaneously and I also find the equilibrium strategies played by the firm which wants to delay the exit of its counterparty. Before any exit occurs, the upstream firm chooses the price of the input sold to the downstream firm and its exit timing in order to maximize its expected present value (EPV). The downstream firm's exit threshold is optimally chosen given the pricing strategy and exit decision of the upstream firm. And the downstream firm also decides the amount of quantities sold to the consumers.

This model builds on several existing strands of literature. Below I review the most relevant literature and discuss the contribution of this paper.

The theory of irreversible investment under uncertainty considers problems in which a firm must choose the optimal timing of investment when the decision cannot be reversed and the value of the project evolves stochastically. The real options approach improves the traditional investment theory[1] by allowing the value of delay and the importance of flexibility to be quantified and incorporated explicitly into the analysis. A thorough review is given in [3].

In reality, many decisions made by firms take into consideration the actions of others, but the basic real-options models do not account for the strategic interaction between firms. However, there is a new strand of literature that incorporates game-theoretic concepts in the real-options framework. Examples of models in discrete time are [15], and [8]. [5, 6], [9, 10, 11], [16], and [12] modeled investment decisions using diffusion processes. [5] and [16] modeled the strategic interactions in leader-follower games under complete information. [2] studies preemption games with irreversible investments under uncertainty. [14] studied irreversible investments under uncertainty in vertical relationships. There are also papers about strategic exit under uncertainty, [10] and [13]. [7] provided a survey of game theoretic real options models.

This paper naturally belongs to this strand of literature. I model uncertainty

[1]The traditional investment theory is based on the rule that an investment project should be undertaken whenever its net present value is positive. However, this decision rule neglects the comparison of the value of investing today and sometime in the future.

using the geometric Brownian motion, which is standard in the continuous-time real-options models. While the majority of this literature considers horizontal competition, I differ from them by considering exit in vertical relationships. In vertical relationships, the firms are not only asymmetric but also rely on the existence of each other. Instead of entry, this paper models the strategic exit in vertical relationships. This paper is mostly related to [13] which adds uncertainty into a model based on [4]. In a vertical relationship, the follower in the exit game suffers from losing its foothold. Since the survival of the firm relies on its counterparty, it is only optimal to exit simultaneously in the structure of both firms are monopolists in its own market.

Most static entry and exit models are generally solved by backward induction starting from the terminal period of the game. But in my stochastic framework, I cannot work backward from a fixed-time moment. I use the state-dependent Markov strategies, which are expressed as stopping sets in the state space such that the firm exits when the state variable hits the corresponding stopping set for the first time. At each state level, the firm has the optimal pricing or quantity strategy that maximizes its life-time expected present value and at the same time there is a corresponding stopping set. The firm exits when the state level falls in its corresponding stopping set for the first time. The equilibrium in my model is subgame perfect.

The contribution of this paper can be seen from two aspects. First, it extends the real-options literature by studying strategic interactions associated with abandonment options in vertical relationships. In most game-theoretic real-options papers, the only strategy the firms decide is the exercise threshold. I enrich the firms strategy space by allowing the firms to decide on price and quantities to maximize their life-time expected present values. Second, this paper studies the interdependence of firms in vertical relationships during hard times whereas best part of the literature concentrate on vertical integration and controls. For both the game-theoretic real-options literature and vertical relationship literature, by introducing uncertainty into the supply chain, this model brings a new view to the interdependence relationship in supply chains. In reality, there exist non-integrated firms in supply chains especially in high technology concentrated industries which have high entry barriers. As mentioned in the three motivation examples, unexpected exogenous shock has been continuously bringing large impact on the vertical relationships in a supply chain. Therefore, this is an important and interesting area to explore.

This paper is organized as follows. In Section 2, I introduce the model. In section 3, I show how the firms' decide their optimal exit timing and derive their best-response strategies given any strategy of their counterparty. In section 4, I analyze the equilibrium in case that the upstream strategically delays the exit of the downstream firm. Section 5 investigates the case in which the downstream firm has the incentive to delay the exit of the upstream firm. Section 6 concludes. All proofs are in the Appendix.

# 2 The Model

This paper considers a declining final good market with an upstream monopolist and a downstream monopolist. In this vertical structure, they are initially not integrated and remain independent of each other. The model is in continuous time with an infinite time horizon. The two firms are labelled U for the upstream firm and D for the downstream firm. Both firms discount the future at rate $\gamma$.

The downstream firm faces a constant price elasticity demand function:

$$D(p_t;X_t) = e^{X_t} p_t^{\delta},$$

where $\delta < -1$ is the constant price elasticity and the demand shock $\{X_t\}_{t\geq 0}$ is a Brownian motion increment defined on a filtered probability space

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t<\infty}, P)$$

satisfying the usual properties[2]. With drift $\mu$ and variance $\sigma^2$, the dynamic of $\{X_t\}_{t\geq 0}$ is expressed by

$$dX_t = \mu dt + \sigma dW_t,$$

and $W_t$ is the standard Wiener process. The inverse demand function is

$$P(q_t;X_t) = \left(q_t/e^{X_t}\right)^{\frac{1}{\delta}}.$$

After observing the demand at time $t$, the downstream firm buys the intermediate good from the upstream firm at price $p_{ut}$ and then sells the good to the consumers. The marginal cost of the upstream firm to produce one unit of the intermediate good sold to the downstream firm is a constant, $mc$.

At the beginning of time $(t = 0)$ when the contract between the two firms is signed, both firms agree that the upstream firm may charge the intermediate good according to a certain price function, say $p_u(X_t)$, and one of the firms may propose

[2]Namely, $\mathscr{F}_0$ contains all the $P-$null sets of $\mathscr{F}$, and filtration $(\mathscr{F}_t)_{0\leq t<\infty}$ is right continuous.

an alternative price function when the time calls[3]. That is to say, the intermediate good price function could only be changed once after the contract is signed. This change in the price function is irreversible [4]. For every time moment, firm U sells the intermediate good to firm D according to the price function agreed in the contract or the new price function. Firm D then decides how much quantity of the final good to sell given the input price and realized demand shock.

Other than deciding when to offer the change in the price function and what new price function to offer, both firms have the option to exit their markets, *i.e.* the firms also have to make an exit decision and this is an irreversible decision. Given the current shock $x$, the strategy of firm $i$, $i = U, D$, is to decide whether to continue operating or to exit or to change the price of the intermediate good. The firms have different strategies, here I explain part of the firms' strategy space and leave the remaining strategy space till later of the paper. Part of the firms' strategies are two stopping sets: when to change the intermediate good price (this may be empty for some firms) and when to exit. A firm's strategy is a stopping rule specifying a threshold or "trigger point" for the stochastic variable $X$ at which the firm exits, *i.e.* firm $i$ chooses an exit threshold $h_i \in \mathbb{R}$ to exit its market, $i = D, U$. To be precise, the statement is that firm $i$ exits the first time when the stochastic process $X_t$ crosses the value $h_i$, crossing this threshold from above. Since the state variable is stochastic, the time when the state variable first crosses an exit threshold is also a random variable. Therefore, instead of choosing a calendar exit time, the firms choose thresholds of the state variable. To summarize, firm $i$ exits if the state variable drops below $h_i$ for the first time and the stopping set, also called the "exit region" in this paper, takes the form of $(-\infty, h_i]$.

At time $t$, both firms observe the realization of the demand shock, $x$. Firm U charges firm D $p_{ut}$ per unit of the intermediate good and firm D decides $q_t$, the amount of the final good to sell. Both firms receive their revenue net of input costs, $\pi_U$ and $\pi_D$ respectively, and pay their operational cost, $F_U$ and $F_D$. The exact expression of $p_{ut}$ and $\pi_i$ will be shown later. After observing the realization of the current shock, both firms decide whether or not to exit. In the vertical structure of this model, when one side of this vertical relationship exits, its counterparty must exit as well. This is the spillover effect of exit in a vertical relationship. The scrap values of both firms are normalized to zero. Should one firm want to exit later than its counterparty, it may be able to delay the counterparty's exit and thus prevent its

[3] For example, when one party wants to exit but the counterparty wishes to stay in the market longer, keeping in mind that the two firms are vertically related and operated independently.

[4] This assumption was made for the sake of tractability. I thank Frank Riedel for this suggestion.

own exit by exercising the option to change the intermediate good price function to $\hat{p}_{ut}(x)$.

At time $t$, given $p_{ut}$ and the realization of the state variable $X_t = x$, firm D maximizes profit at time $t$ by solving the following problem:

$$\max_{q_t} \quad q_t(p - p_{ut}) - F_D, \tag{2.1}$$

where $p(q_t;x) = (q_t/e^x)^{1/\delta}$. Firm $i$ incurs a fixed operating cost $F_i$, $i = D,U$, at each moment in time if firm $i$ is in business at time $t$. By solving (2.1), firm D maximizes its profit at

$$q_t = \left(\frac{\delta}{\delta+1}\right)^{\delta} p_{ut}^{\delta} e^x. \tag{2.2}$$

Firm U takes (2.2) as given and maximizes its current profit:

$$\max_{p_{ut}} \quad (p_{ut} - mc)\cdot q_t - F_U. \tag{2.3}$$

The price that maximizes (2.3) is

$$\bar{p}_{ut} = \bar{u}\cdot mc, \tag{2.4}$$

where $\bar{u} = \delta/(\delta+1)$. One can observe that though the quantity of the final good is state dependent, the price of the intermediate good which maximizes firm U's time t profit, $\bar{p}_{ut}$, is a constant and is independent of the state variable, $X_t$[5].

**Lemma 2.1.** *At time $t$, given the realization of the stochastic state variable, $x$, the firms' time t profits are are maximized by*

$$\bar{p}_{ut} = \bar{u}\cdot mc \quad \textit{and} \quad \bar{q}_t = \left(\frac{\delta}{\delta+1}\right)^{\delta} (\bar{u}\cdot mc)^{\delta} e^x, \tag{2.5}$$

*where $\bar{u} = \delta/(\delta+1) > 1$ with $\delta < -1$.*

As I mentioned earlier in this section, the firms have an irreversible option to change the price function from the original price function specified in the contract,

[5] If firm U produces the intermediate good with a decreasing return to scale production function with a single input $y$, for example $f(y) = y^{\alpha}$ and $\alpha < 0$, then the $p_{ut}$ that maximizes firm U's time t profit would be state dependent. The model of this paper is a special case with $\alpha = 1$.

$p_u(u, X_t)$ to another price function, $\hat{p}_u(u, X_t)$. For simplicity, it is assumed that the price functions take the form $p_u = p_u(u, X_t) = u \cdot mc$, where $u \in [1, \infty)$. The markup $u$ is greater or equal to 1 because it is not reasonable for firm U to sell the intermediate good at a price lower than its marginal cost. With $p_u = u \cdot mc$ and $u \in [1, \infty)$, the firms will receive profits of the form

$$\pi_i(u, x) = W_i(u)e^x - F_i, \quad i = D, U, \tag{2.6}$$

where

$$W_U(u) = (u-1)u^{\delta} mc^{\delta+1} \left( \frac{\delta}{\delta+1} \right)^{\delta} \quad \text{and,} \tag{2.7}$$

$$W_D(u) = \left( \frac{-1}{\delta+1} \right) (u \cdot mc)^{\delta+1} \left( \frac{\delta}{\delta+1} \right)^{\delta}. \tag{2.8}$$

Under the price signed in the contract, there could be three possible cases: (i) both firms want to exit simultaneously; (ii) firm D wants to exit before firm U and (iii) firm U wants to exit before firm D.

To formalize the objective functions of the optimal exit timing for both firms, I define $\tau_k = inf\{t > 0 : X_t \leq h_k(X_t)\}$ as the time when the state variable hits the exit region of firm $k$ for the first time, before any price change. Let firm $i$ be the first firm which reaches its exit threshold under the price function specified in the contract, and let firm $j$ be the firm which wants to stay in the market longer. Given the strategies of firm $i$, firm $j$ chooses when to change the price ($h_s$) and what new price to change ($\hat{u}$) , i.e. firm $j$ offers to change $p_u$ to $\hat{p_u}$ when the stochastic state falls below $h_s$ for the first time. Firm $j$ also chooses its own exit threshold $h_j$ to maximize its value at the current state, $X_t = x$. Firm $j$'s maximization problem:

$$\begin{aligned} V_j(u, x) = \quad & \max_{\tau_s, \hat{\tau}_j, \hat{u}} \mathbb{E}^x \left[ \int_0^{\tau_s} e^{-\gamma t} \pi_j(u, x) dt \right] + \mathbb{E}^x \left[ \int_{\tau_s}^{\hat{\tau}_j} e^{-\gamma t} \pi_j(\hat{u}, x) dt \right] \\ & \text{s.t.} \quad h_s \geq h_i \geq \hat{h_j} \geq \hat{h_i} \end{aligned} \tag{2.9}$$

where $\hat{u}$ means changing the intermediate good price from $p_u = u \cdot mc$ to $\hat{p_u} = \hat{u} \cdot mc$. The threshold of when to change the price is $h_s$ and $\tau_s = inf\{t > 0 | X_t \leq h_s\}$. Firm $i$'s and firm $j$'s new exit thresholds under the new price function are $\hat{h_i}$ and $\hat{h_j}$ respectively and $\hat{\tau}_i = inf\{t > 0 | X_t \leq \hat{h_i}\}$ and $\hat{\tau}_j = inf\{t > 0 | X_t \leq \hat{h_j}\}$. Firm $j$ must offer the new price before the exit of firm $i$, $h_s \geq h_i$, otherwise firm $i$ will exit and so firm $j$ will have to exit.

Since firm $j$ is not ready to exit at the threshold that firm $i$ wants to exit under the price signed in the contract, firm $j$ will propose the new price, $\hat{p}_u$, at the switching threshold $h_s \geq h_i$. Thus firm $i$, the firm which originally has a higher exit threshold, takes the switching threshold $h_s(\tau_s)$ as given. Firm $i$ solves the following problem:

$$V_i(u,x) = \max_{\tau_i, \hat{\tau}_i} \quad \mathbb{E}^x \left[ \int_0^{\tau_s \wedge \tau_i} e^{-\gamma t} \pi_i(u,x) dt \right] + 1_{\{\tau_s \geq \tau_i\}} \mathbb{E}^x \left[ \int_{\tau_s}^{\hat{\tau}_i} e^{-\gamma t} \pi_i(\hat{u},x) dt \right]. \tag{2.10}$$

Here in the maximization problem of both firms, I no longer put $q_t$ nor $p_u$ as choice variables, because the best response of $q$ given $p_u$ and $p_u = u \cdot mc$ are already plugged into the profit function $\pi_k(\cdot)$. Therefore, in this objective function of firm $i$, firm $i$ only chooses the optimal exit threshold, given the strategies of firm $j$,.

The expected present value (EPV) of the flow $\mathbb{E}[\int_0^\infty e^{-\gamma t} W_i(u) e^{X_t} dt]$ is finite iff $\mathbb{E}[e^{X_t}] < \infty$ and the no-bubble condition $\gamma - \Psi(1) > 0$ holds. Here $\Psi$ is the Lévy exponent of the Brownian motion definable from $\mathbb{E}[e^{AX_t}] = e^{t\Psi(A)}$. Indeed, if $\Psi(1) < \gamma$, then by Fubini's theorem,

$$\mathbb{E}^x \left[ \int_0^{+\infty} e^{-\gamma t} W_i(u) e^{X_t} dt \right] \equiv \mathbb{E} \left[ \int_0^{+\infty} e^{-\gamma t} W_i(u) e^{X_t} dt | X_0 = x \right]$$
$$= \int_0^{+\infty} e^{-\gamma t} W_i(u) \mathbb{E}^x[e^{X_t}] dt = W_i(u) \int_0^{+\infty} e^{-\gamma t + z + t\Psi(1)} dt = \frac{W_i(u) e^x}{\gamma - \Psi(1)}.$$

The value functions are well-defined if and only if

$$\gamma - \Psi(1) > 0, \tag{2.11}$$

where $\Psi(z) = \mu z + \frac{\sigma^2}{2} z^2$. This is the no-bubble condition for the value functions.

# 3 Value Functions and Thresholds

To solve the firms' problem in (2.9) and (2.10), it is important to understand the behavior of the firms' thresholds and profit functions. So far, I have not yet discussed what is the optimal price to sign in the contract. Though from Lemma 2.1 it is natural and tempting to think that the price specified in the contract should be $\bar{p_u} = \bar{u} \cdot mc$, I will show that this is only true under a certain condition in this section.

## 3.1 Optimal Exit Timing without Changing the Price Function

In this subsection, I investigate which firm would want to exit first given that there is no change in the price function. Since there is nothing the firms could do to delay the exit of their counterparty, it is simply the standard real options problem applied to exit decisions. The firms first calculate their own exit threshold as if independent of their counterparty, then they see who wants to exit first. The firm who wants to exit later would have to exit with the firm who will exit earlier in this framework. The optimal exit timing of both firms is derived by solving the following system of equations, $i = D, U$:

$$\begin{aligned} \left(\gamma - \mu\frac{\partial}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\right) V_i(u,x) &= W_i(u)e^x - F_i && \text{if } x > h_i \\ V_i(u,x) &= 0 && \text{if } x \leq h_i \end{aligned}$$

The above second order differential equation is a standard real-options problem, please refer to [3] for details. The closed form solution to $h_i$ is

$$e^{h_i} = \frac{F_i}{\kappa^+(1)W_i(u)}, \tag{3.1}$$

where $\kappa^+(1) = \beta^+/(\beta^+ - 1)$ and $\beta^\pm$ are the roots of $\gamma - \Psi(1) = 0$. By the no bubble condition in (2.11), $\beta^+ > 1 > 0 > \beta^-$. Please note that the firms are considering strategic interactions when it comes to making profits for every time moment $t$ but are nonstrategic about exit timing in this section.

**Theorem 3.1.** *Given that $p_{ut} = u \cdot mc$, $\forall t > 0$,*

1. *both firms would exit simultaneously ($h_D = h_U$) iff $\frac{F_D}{F_U} = \Delta \cdot \left(\frac{u}{u-1}\right)$;*
2. *firm D would want to exit before firm U ($h_D > h_U$) iff $\frac{F_D}{F_U} > \Delta \cdot \left(\frac{u}{u-1}\right)$, and*
3. *firm U would want to exit before firm D ($h_D < h_U$) iff $\frac{F_D}{F_U} < \Delta \cdot \left(\frac{u}{u-1}\right)$,*

*where $\Delta = -1/(\delta + 1)$.*

**Theorem 3.2.** *If the price signed in the contract is such that $\frac{F_D}{F_U} = \Delta \cdot \left(\frac{u}{u-1}\right)$, both firms exit simultaneously at $h^* = h_U = h_D$ and the option to change the intermediate good price function is not exercised.*

Apparently, the last two cases are more interesting. It is tempting to ask what would happen if one side of the supply chain wishes to exit later than its upstream or downstream? Is there any thing it could do to delay the exit of its counterparty and prevent its own exit? If yes, how long could the firm postpone the exit? Before answering these question, one must understand how the exit thresholds of the firms behave.

**Theorem 3.3.** $\forall u \in [1,\infty)$, $\bar{u} = \underset{u}{\operatorname{argmin}} \quad h_U(u)$.

**Theorem 3.4.** *$h_D(u)$ is strictly decreasing in u.*

The deterministic component of firm U's profit $W_U(u)$ is concave in $u < (\delta - 1)/(\delta+1)$ and is globally maximized at $\bar{u}$, therefore the lowest exit threshold possible for firm U is $h_U(\bar{u})$. Whereas the deterministic component of firm D's profit $W_D(u)$ is strictly decreasing in $u$. When $u$ becomes higher, the cost for firm D increases. Please refer to the appendix for details.

**Theorem 3.5.** *If $p_{ut} = \bar{u} \cdot mc$, $\forall t > 0$, then*

1. *both firms would exit simultaneously ($h_D = h_U$) iff $\frac{F_D}{F_U} = \bar{u}$;*
2. *firm D would want to exit before firm U ($h_D > h_U$) iff $\frac{F_D}{F_U} > \bar{u}$, and*
3. *firm U would want to exit before firm D ($h_D < h_U$) iff $\frac{F_D}{F_U} < \bar{u}$.*

## 3.2 The Optimal Switching Time

One interesting characteristic of a vertical relationship is that the two asymmetric firms not only compete in how to share the pie of the final good market, but also rely on the existence of each other. When one party exits, its counterparty must exit as well. Following the previous section, let firm $i$ be the first firm which reaches its exit threshold under the price function specified in the contract, and firm $j$ be the firm which wants to stay in the market longer, *i.e.* $h_i \geq h_j$.

Firm $j$ would want the new price to be such that both firms exit after the price change and itself exits no later than firm $i$. By changing the price to delay the exit of its counterparty, firm $j$ is sacrificing its own value to subsidize the other firm so that it could be in the market longer. Thus given that the price signed in the contract is optimal, it is never optimal to change the price too early as firm $j$ is better off by delaying the change until $h_i$. However, if firm $j$ waits after

$x < h_i$ to change the price function, firm $i$ would have already exit at $x = h_i$ (exit is irreversible) and firm $j$ will have to exit at the same time. Hence in order to prevent firm $i$'s exit, firm $j$ maximizes (2.9) by $h_s = h_i$.

## 3.3 The Optimal New Price

Since $h_s = h_i$ and $h_i$ is independent of $\hat{u}$, the first integral in (2.9) is independent of $\hat{u}$. This leaves firm $j$ to decide the level of $\hat{u}$ and the optimal exit threshold after the price change to maximize the second integral in (2.9). Firm $j$ solves the following problem:

$$\hat{V}_j(\hat{u},x) = \max_{\hat{u},\hat{\tau}_j} \quad \mathbb{E}^x\left[\int_{\tau_s}^{\hat{\tau}_j} e^{-\gamma t}\pi_j(\hat{u},x)dt\right] \quad \text{s.t.} \quad \hat{h_j} \geq \hat{h}_i. \tag{3.2}$$

The expected present value of receiving the new profit until firm $j$'s exit at $\hat{h_j}$ is equivalent to the following:

$$\hat{V}_j(\hat{u},x) = \max_{\hat{u},\hat{\tau}_j} \quad -\mathbb{E}^x\left[\int_0^{\tau_s} e^{-\gamma t}\pi_j(\hat{u},x)dt\right] + \mathbb{E}^x\left[\int_0^{\hat{\tau}_j} e^{-\gamma t}\pi_j(\hat{u},x)dt\right]. \tag{3.3}$$

Given $\hat{u}$, I obtain the optimal exit threshold $\hat{h_j}(\hat{u})$ by maximizing the second integral in (3.3). From [1],

$$\mathbb{E}^x\left[\int_0^{\hat{\tau}_j} e^{-\gamma t}\pi_j(\hat{u},x)dt\right] = \frac{1}{r}\varepsilon^{-}1_{(\hat{h_j},\infty)}\varepsilon^{+}[W_j(\hat{u})e^x - F_j], \tag{3.4}$$

where $\hat{h_j}$ is such that $\varepsilon^{+}[W_j(\hat{u})e^x - F_j] = 0$. To be more exact, $e^{\hat{h_j}} = \frac{F_j}{\kappa^{+}(1)W_j(\hat{u})}$.

Firm $i$ takes $\hat{u}$ as given, and it solves the exit problem in (3.3). Thus, firm $i$'s optimal exit threshold is $e^{\hat{h_i}} = \frac{F_i}{\kappa^{+}(1)W_i(\hat{u})}$.

Therefore $\hat{h_U} \geq \hat{h_D}$ iff

$$W_D(\hat{u}) \geq \frac{F_D}{F_U}W_U(\hat{u}). \tag{3.5}$$

**Lemma 3.6.** *$W_D(\hat{u}) = \frac{F_D}{F_U}W_U(\hat{u})$ has an unique solution $\hat{u}^*$, $\forall \frac{F_D}{F_U} > 0$.*

**Lemma 3.7.** *$\hat{u}^* < \bar{u}$ if $\frac{F_D}{F_U} > \bar{u}$ and $\hat{u}^* \geq \bar{u}$ if $\frac{F_D}{F_U} \leq \bar{u}$.*

Ideally, firm $j$ would want to change the price to $\hat{u}^*$ so that its own exit threshold increases and firm $i$'s exit threshold decreases. It is very intuitive that firm $j$ will want to change the contract price to a price such that it can exit at its optimal exit threshold under the new price. According to Theorem 3.3 and Theorem 3.4, for $u > \bar{u}$, both firms are worse off. Hence the maximum possible price of the intermediate is $\bar{u}$, *i.e.* $u, \hat{u} \in [1, \bar{u}]$.Due to the asymmetry between the two firms and the initial price in the contract, the change in price may not occur. This will become clear in the next two sections of this paper. In the following two sections, case (ii) and case (iii) of Theorem 3.1 will be discussed respectively.

# 4 The Upstream Strategically Delays the Exit of the Downstream Firm

This section investigates the case that if there is no change in the price function, under $p_u = u \cdot mc$, firm D would want to exit earlier than firm U, therefore forcing firm U to exit as well. Firm U strategically delays the exit of firm D. The optimal contract price in this case is found.

Firm D solves (2.10) taking the intermediate good price function change threshold of firm U into consideration. Since the best response of $q$ given $p_u$ is already plugged into $\pi_D(\cdot)$, it is only left for firm D to decide when to exit optimally for a given price schedule $\hat{p}_u$. As shown in (2.10),

$$\text{(4.1)} \qquad \max_{\hat{\tau_D}} \quad \mathbb{E}^x \left[ \int_0^{\tau_s} e^{-\gamma t} \pi_D(u,x) dt \right] + \mathbb{E}^x \left[ \int_{\tau_s}^{\hat{\tau_D}} e^{-\gamma t} \pi_D(\hat{u},x) dt \right].$$

Firm D solves (4.1) when it exits after the price change. As discussed in the previous section, the first expectation is independent of $\hat{\tau_D}$, so firm D only needs to find the $\hat{\tau_D}$ that maximizes the second expectation. The second expectation can be taken as an option to acquire payoff stream $\pi_D(\hat{u}, x)$ at $\tau_s$ with the option to exit at $\tau_D$. Both firms know that $\tau_s = \tau_D$. Firm D threating to exit earlier than $h_D$ under the initial price is not credible, because exiting earlier than $h_D$ is not optimal. Also, it makes no sense for firm D to exit later than its optimal exit threshold. If firm D exits after its optimal exit threshold, firm U will take the advantage to postpone the price change. So the only thing left for firm D to do is to choose the optimal timing to abandon the payoff stream $\pi_D(\hat{u}, x)$, which is the $h_D(\hat{u})$ derived in Section 3.3.

## 4.1 The Optimal Prices

From Section 3.3, firm U will want to change the new price to $\hat{u}^*$. Firm U maximizes (2.9) given $h_D(\hat{u}^*)$ and $\hat{u}^*$ . Firm U's value function, $V_U(u,x)$, can be expressed as a function of $u$:

$$V_U(u,x) = \frac{W_U(u)e^x}{\gamma-\Psi(1)} - \frac{F_u}{\gamma} + \left[\frac{F_u}{\gamma} - \frac{W_U(u)e^{\hat{h}_U(\hat{u}^*)}}{\gamma-\Psi(1)}\right]\left(\frac{e^{\hat{h}_U(\hat{u}^*)}}{e^x}\right)^{-\beta^-}$$

$$-\frac{[W_U(u)-W_U(\hat{u})]e^{h_s}}{\gamma-\Psi(1)}\left(\frac{e^{h_s}}{e^x}\right)^{-\beta^-} . \tag{4.2}$$

In order to find the price $u$ which maximizes $V_U(u,x)$, one has to study the behavior of $V_U(u,x)$ as a function of $u$ with $x$ fixed. Let $\bar{V}_U = V_U(\bar{u},x)$ and $V_U = V_U(u,x)$. Then

$$\bar{V}_U - V_U = \frac{[W_U(\bar{u})-W_U(u)]e^x}{\gamma-\Psi(1)} + \frac{e^{\beta^- x}}{\gamma-\Psi(1)}\left\{-[W_U(\bar{u})-W_U(\hat{u})]e^{\bar{h}_D(-\beta^-+1)} + [W_U(u)-W_U(\hat{u})]e^{h_D(-\beta^-+1)}\right\}, \tag{4.3}$$

where $e^{\bar{h}_D} = \frac{F_D}{\kappa^+(1)W_D(\bar{u})}$. Please keep in mind that $u \le \bar{u}$. Also, firm U's time $t$ profit is maximized at $\bar{u}$ and firm D's time $t$ profit is strictly decreasing in $u$. Thus the first term of $\bar{V}_U - V_U$ is positive and the second term is negative. The sign of $\bar{V}_U - V_U$ depends on the value of $x$. If $x \nearrow +\infty$, then the second term of $(\bar{V}_U - V_U) \nearrow 0$ and $\bar{V}_U - V_U \nearrow \infty$. However, if $x \searrow \bar{h}_D$, $\bar{V}_U - V_U < 0$. This is very intuitive. For $x \searrow \bar{h}_D$, firm U does not have that many days left to receive $\pi_U(\bar{u},x)$ and needs to shift to a lower profit flow, $\pi_U(\hat{u}^*,x)$, soon. Thus it would be better off by setting the initial price at $u < \bar{u}$ and receive $\pi_U(u,x) > \pi_U(\hat{u}^*,x)$ for a longer while. When the initial state is so high that $(\bar{V}_U - V_U) > 0$ for all $u \in (0,\bar{u})$, firm U will set the price in the contract as $\bar{u}$ because the time to change the price is still far away. For the simplicity of solving the equilibrium of this game, in the rest of this paper, the initial state variable $x_0$ is assumed to sufficiently large and the contract price is indeed the price that maximizes firm U's time $t$ profit, $\bar{u}$.

## 4.2 Equilibrium when Firm D Wants to Exit Earlier than Firm U

By proposing any $\hat{u}$ that satisfies (3.5) would make firm D have smaller exit threshold than firm U. Recall from Theorem 3.5 that the case discussed in this section occurs iff $\frac{F_D}{F_U} > \bar{u}$, which makes $\hat{u}^* < \bar{u}$. Since $W_U(\hat{u})$ is increasing in $\hat{u}$ for $\hat{u} \leq \bar{u}$, it is maximizing firm U's value function to have $\hat{u}$ set as making the equality sign hold in (3.5). Since $W_D(\cdot)$ is strictly decreasing in $u$, by lowering the price from $\bar{u}$ to $\hat{u}^*$ lowers the exit threshold of firm D and therefore delays the exit of firm D.

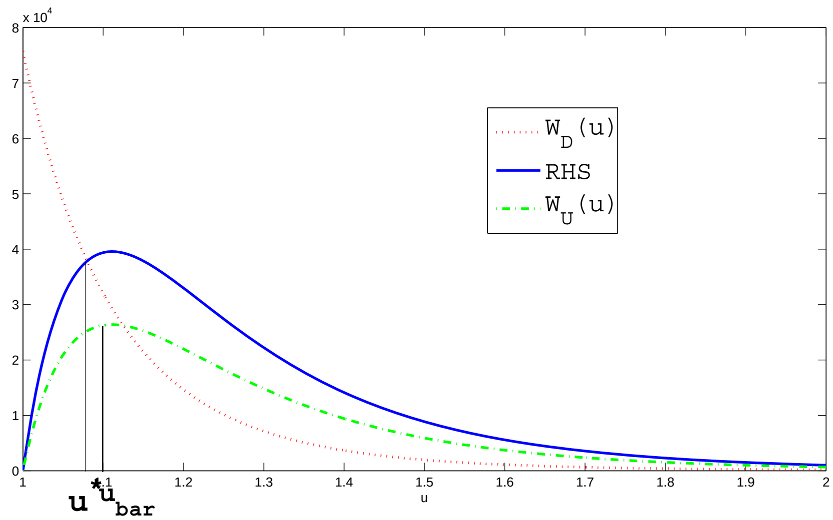


Figure 4.1: Case 2: $h_D > h_U$

**Theorem 4.1.** *There is a unique intermediate good price function*

$$p_{ut}(\hat{u^*}) = \hat{u}^* \cdot mc < \bar{p_u} \tag{4.4}$$

*such that both firm optimally exit simultaneously.*

When there is no change in the price function, the downstream firm initially wishes to exit prior to the optimal exit time of the upstream firm. Hence the upstream firm will exercise the option to make an one time change in the price function from $p_{ut} = \bar{u} \cdot mc$ to $\hat{p_{ut}} = \hat{u}^* \cdot mc$ when the realization of the state variable first crosses the downstream firm's initial exit threshold $h_D$ from above. With the new price function, it is optimal for both firms to exit simultaneously.

**Theorem 4.2.** *Assume that the initial state is sufficiently high. When the downstream firm initially wishes to exit prior to the upstream, the unique equilibrium is that both firms receive the maximized profit until $h_s$ and then firm U will propose to* *lower the intermediate price so that it is optimal for both firms exit simultaneously at $h^*$.*

$$u^* = \begin{cases} \bar{u} & if \quad x > h_s, \\ \hat{u}^* & if \quad x \leq h_s \end{cases}, \quad h_s = ln\left(\frac{F_D}{\kappa^+(1)W_D(\bar{u})}\right), \quad and \quad \hat{h_U} = \hat{h_D} = h^*,$$

*where $\bar{u} > \hat{u}^*$ and $h^* = \frac{F_U}{\kappa^+(1)W_U(\hat{u})}$.*

## 4.3 Expected Delay in Exit

With the option to change the price function once, how much time could the upstream firm postpone the exit? An analytical expression for expected waiting time with a Brownian motion as the historic measure is derived. Assume that the current state $x > h^*$ and consider the waiting time $\tau^*$ till the option to exit will be exercised. This is a random variable defined by $\tau^* = min\{t > 0 | X_t \leq h^*\}$.

**Lemma 4.3.** *For $X_0 = x > h^*$, the expected waiting time (under the historic measure $\mathbb{P}$ of a Brown motion with drift $\mu < 0$ and variance $\sigma^2$) can be calculated as follows:*

$$\mathbb{E}_{\mathbb{P}}[\tau_x^*] = \frac{1}{\mu}(h^* - x).$$

**Theorem 4.4.** *At the date firm U change the intermediate good price to $\hat{p_u}$, it knows the expected delay in exit conditioned on the state level of that date is*

$$\mathbb{E}_{\mathbb{P}}[\tau_{h_s}^*] = \frac{1}{\mu}(h^* - h_s) = \frac{1}{\mu} ln\left[\frac{F_U W_D(\bar{u})}{F_D W_U(\hat{u})}\right].$$

# 5 The Upstream Firm Initially Exits Before the Downstream Firm's Optimal Exit Time.

This an interesting part, as now the roles of the firms are switched, however the result is not necessarily the same as the previous case with the subscript switched. Given the strategy space in this paper, the downstream may not do anything to delay the exit of the upstream firm.

In this case, firm U is the follower who takes the time of price change and the new price as given. Taking $h_s(\tau_s)$ and $\hat{u}$ as given, Firm U's maximization problem is

$$\max_{\hat{\tau_U}} \mathbb{E}^x\left[\int_0^{\tau_s} e^{-\gamma t}\pi_U(x,\bar{u})dt\right] + \mathbb{E}^x\left[\int_{\tau_s}^{\hat{\tau_U}} e^{-\gamma t}\pi_U(\hat{u},x)dt\right]. \tag{5.1}$$

Firm D needs to solve the following problem:

$$\begin{aligned} \max_{\tau_s,\hat{\tau_D},\hat{u}} \quad & \mathbb{E}^x\left[\int_0^{\tau_s} e^{-\gamma t}\pi_D(x,\bar{u})dt\right] + \mathbb{E}^x\left[\int_{\tau_s}^{\hat{\tau_D}} e^{-\gamma t}\pi_D(\hat{u},x)dt\right] \\ \text{s.t.} \quad & h_s \geq \hat{h_D} \geq \hat{h_U} \end{aligned} \tag{5.2}$$

Before actually solving (5.1), it would be interesting to think about Theorem 3.3 again. Theorem 3.3 states that the lowest exit threshold possible for firm U is $h_U(\bar{u})$ . If the initial contract price is some $u < \bar{u}$ and results in this case discussed in this section, $h_U > h_D$. Then by Theorem 3.3, the best firm D could do to delay the exit of firm U is by increasing the intermediate good price to $\bar{u}$. However, if firm U is going to exit at $\bar{h_U}$ eventually, then it will not agree to sign the contract with $u < \bar{u}$. It will only sign the contract with price $\bar{u}$. If the firms are in Case (iii) of Theorem 3.5, then there is nothing that firm D can do but to exit simultaneously with firm U at $\bar{h_U}$ and the option to change the price is useless. Unless there is a modification to the strategy space such that the firms strategy space is enriched so that the firms can adjust price every $t$. But this will make firms strategies become intractable.

**Theorem 5.1.** *When the upstream firm initially wishes to exit prior to the downstream firm, there is nothing the downstream firm can do to delay the exit of the upstream firm. Therefore the unique equilibrium is that both firms exit simultaneously at the upstream firms exit threshold.*

$$u^* = \bar{u} \quad \forall x, \quad \textit{and} \quad h^* = ln\left(\frac{F_U}{\kappa^+(1)W_U(\bar{u})}\right).$$

However, should the firms have only the exit timing to decide and there is no additional option to change the price, it is true that there exist an unique price, $\tilde{p_u} = \tilde{u}\cdot mc$, such that both firms will exit simultaneously. By Lemma 3.7, $\tilde{u} \geq \bar{u}$.

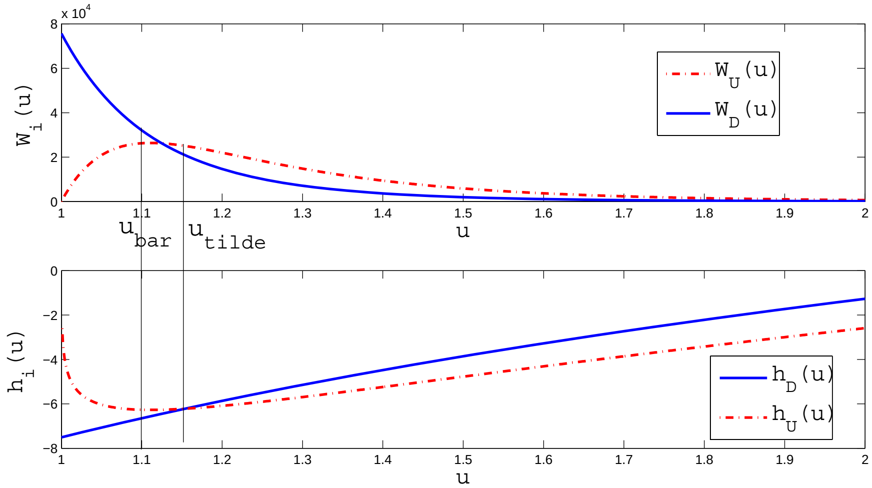


Figure 5.1: Case 3: $h_U > h_D$

# 6 Conclusion

This paper studies the strategic delay of exit in a vertical relationship of a supply chain in a stochastically declining downstream market. The firms not only decide when to permanently exit their markets respectively but also make quantity and price decisions. I have shown how strategic interactions change the firms exit timing and how a firm which wants to exit later strategically delays the exit of its counterparty and thus prevents its own exit. There exists a unique equilibrium in which both firms in the end exit simultaneously.

There could be three possible cases for the ordering of non-strategic exit thresholds: (i) both firms exit simultaneously; (ii) the downstream firm exits prior to the upstream firm; (iii) the upstream firm exits before the downstream firm. At the beginning of time, both firms agree to the intermediate good price function that maximizes their profits for every time moment $t$. Since the firms have an option to make one time change to the intermediate good price function in the future, this is naturally the optimal strategy for both firms as they are profit maximizing up to the date a non-strategic exit threshold is reached. In case (i), both firms optimally exit simultaneously at their exit thresholds without the price change option being exercised. When the last two cases occur, the firm which has a lower optimal exit threshold has the incentive to strategically delay its counterparty's exit, and therefore prevent its own exit. In case (ii), the upstream firm will propose to lower the price of the intermediate good once the higher non-strategic exit threshold is reached. This new price function is such that that they would both exit simultaneously later at an optimal time under the new price function. However, in case (iii), though the downstream firm would like to delay the exit, it cannot not do anything to prevent the exit. Therefore, the downstream firm would exit right after the exit of the upstream firm. In the example of Honda and its upstream firms during the natural catastrophes in year 2011, Honda could not have delayed the exit of its upstream firms if they wanted to exit. But Honda's upstream firms could have delayed the exit of Honda if Honda wanted to exit.

From Theorem 4.1, there exists a unique $\hat{u}$ such that both firms would both optimally exit simultaneously. If the firms strategy space is restricted to only deciding an optimal exit time given a price function signed in the contract, in a vertical relationship, there could be no strategic delaying. And the firm which is not ready to exit would have to exit prior to its optimal exit time. Of course the firms could strategically decide the price function in the contract to maximize their lifetime profit, however, a question of bargaining power would arise. In order to exploit efficiency, it is important that the firms are given flexibility to make offers to strategically delay its counterparty's exit and therefore prevent its own exit. The firms could maximize their lifetime value by maximizing their profits every time moment $t$ until a price change occurs and continue to make profits until the exit occurs. The exit is optimally chosen under the new price and with this additional option that allows the firms to make an one time change in the price function increases the firms' value. The strategy space in this paper is restricted to the firms can only strategically change the price once. The problem would become more interesting if the firms could choose their price and quantity for every time moment $t$ to maximize their lifetime expected value, but mathematically this problem becomes intractable.

The model in this paper gives the fundamental results of strategic exit in vertical relationships under uncertainty and could be extended in several dimensions. One interesting dimension is to increase competition on one side of the vertical relationship and see how this would affect the exit decisions. Another interesting future research would be increasing the number of downstream monopolists in disconnected regions who all buy from the upstream monopolist and see how multiple exits in the downstream market will lead to the exit of the upstream firm. It is uncertain if the upstream firm would strategically delay the exit of one single downstream firm. To improve the double mark-up issue in this paper, a variation of this model could be done by exploiting a joint monopolist profit and let the firms bargain on how they would split the big pie.

## References


[1] Svetlana Boyarchenko and Sergei Levendorskiǐ. *Irreversible Decisions under Uncertainty: Optimal Stopping Made Easy*. Springer, 2007.

[2] Svetlana Boyarchenko and Sergei Levendorskiǐ. Preemption games under levy uncertainty. Available at SSRN: http://ssrn.com/abstract=1841823 or http://dx.doi.org/10.2139/ssrn.184182, 2011.

[3] A. Dixit and R. Pindyck. *Investment Under Uncertainty*. Princeton University Press, Princeton, NJ, 1994.

[4] Pankaj Ghemawat and Barry Nalebuff. Exit. *RAND Journal of Economics*, 16(2):184–194, Summer 1985.

[5] Steven R Grenadier. The strategic exercise of options: Development cascades and overbuilding in real estate markets. *Journal of Finance*, 51(5):1653–79, December 1996.

[6] Steven R. Grenadier. Option exercise games: An application to the equilibrium investment strategies of firms. *Review of Financial Studies*, 15(3):691–721, 2002.

[7] K J M Huisman, P M Kort, G Pawlina, and J J J Thijssen. Strategic investment under uncertainty: a survey of game theoretic real option models. *Journal of Financial Transformation*, 13:pp. 111–118, 2005.

[8] Nalin Kulatilaka and Enrico C. Perotti. Strategic growth options. *Management Science*, 44(8):1021–1031, August 1998.

[9] B M Lambrecht. Strategic sequential investments and sleeping patents. *In: Project Flexibility, Agency, and Product Market Competition: New Developments in the Theory and Application of Real Options Analysis. Oxford University Press, Oxford*, pp. 297-323, 2000.

[10] Bart M Lambrecht. The impact of debt financing on entry and exit in a duopoly. *Review of Financial Studies*, 14(3):765–804, 2001.

[11] Bart M. Lambrecht. The timing and terms of mergers motivated by economies of scale. *Journal of Financial Economics*, 72(1):41–62, April 2004.

[12] Robin Mason and Helen Weeds. Can greater uncertainty hasten investment? *Working Papers*, 2006.

[13] Pauli Murto. Exit in duopoly under uncertainty. *RAND Journal of Economics*, 35(1):111–127, Spring 2004.

[14] Richard Ruble, Bruno Versaevel, and tienne de Villemeur. Timing vertical relationships. TSE Working Papers 10-181, Toulouse School of Economics (TSE), June 2010.

[15] H.T.J Smit and L.A. Ankum. A real options and game-theoretic approach to corporate investment strategy under competition. *Financial management*, 22:241–250, 1993.

[16] Helen Weeds. Strategic delay in a real options model of r&d competition. *Review of Economic Studies*, 69(3):729–47, July 2002.

# A Proof of Lemma 3.6 and Lemma 3.7

The shape of $W_D(u)$:

(A-1)
$$W_D(u) = \frac{-1}{\delta+1}\left(\frac{\delta}{\delta+1}\right)^{\delta}(u\cdot mc)^{\delta+1}$$

(A-2)
$$\frac{dW_D}{du} < 0 \quad \text{and} \quad \frac{dW_D^2}{du^2} > 0 \quad \forall u \in [1,\infty)$$

The shape of $W_U(u)$:

$$W_U(u) = (u-1)\left(\frac{\delta}{\delta+1}\right)^{\delta}(mc)^{\delta+1}u^{\delta}$$

$$\frac{dW_U}{du} = \left(\frac{\delta}{\delta+1}\right)^{\delta} mc^{\delta+1}\left\{(\delta+1)u^{\delta} - \delta u^{\delta-1}\right\}$$

$$\begin{cases} > 0 & \text{if} \quad u < \frac{\delta}{\delta+1}, \\ \leq 0 & \text{if} \quad u \geq \frac{\delta}{\delta+1} \end{cases}$$

$$\frac{dW_U^2}{du^2} = \left(\frac{\delta}{\delta+1}\right)^{\delta} mc^{\delta+1}\left\{\delta(\delta+1)u^{\delta-1} - \delta(\delta-1)u^{\delta-2}\right\}$$
$$\begin{cases} > 0 & \text{if} \quad u < \frac{\delta-1}{\delta+1}, \\ \leq 0 & \text{if} \quad u \geq \frac{\delta-1}{\delta+1} \end{cases}$$

$W_D(\cdot)$ is strictly decreasing in $u$ and is converging to zero as $u$ converges to positive infinity. $W_U(\cdot)$ is concave for $u \leq (\delta-1)/(\delta+1)$, convex for $u > (\delta-1)/(\delta+1)$ and is global maximized at $\bar{u} = \delta/(\delta+1)$. The limits of $W_U(\cdot)$ converge to zero as $u$ converges to one and positive infinity. Also, $W_D(\cdot)$ is greater than $W_U(\cdot)$ for all $u > (\delta+1)/(\delta+2)$. And $\frac{\delta-1}{\delta+1} > \frac{\delta+1}{\delta+2} > \frac{\delta}{\delta+1}$. Therefore, $W_D(u) = W_U(u)$ has an unique solution at $u = \frac{\delta+1}{\delta+2}$.

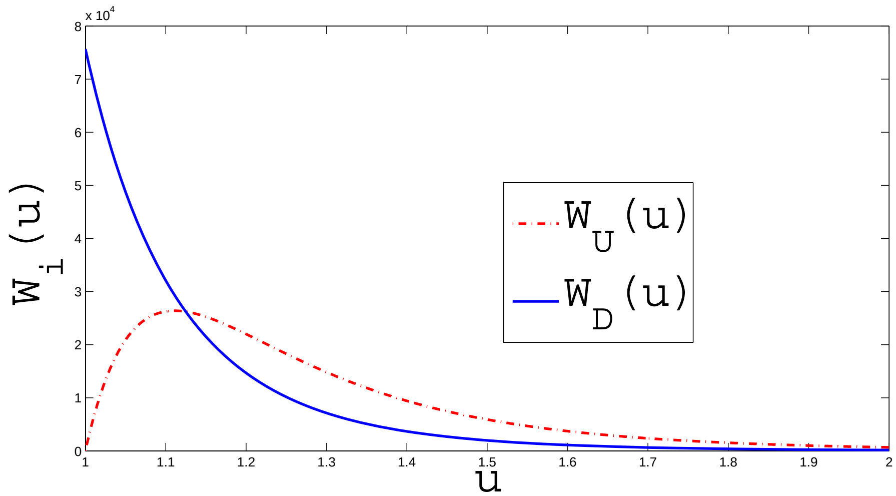


Figure A.1: The shape of $W_U$ and $W_D$

(A-3) also has an unique solution. The RHS of (A-3) has the same shape and properties as $W_U(\cdot)$ as it is only multiplied by a constant.

(A-3) $$W_D(u) = \frac{F_D}{F_U} W_U(u)$$

From Lemma 3.6, it can be derived that $\hat{u}^* = \frac{F_D}{F_D + F_U\left(\frac{1}{\delta+1}\right)}$. And $\hat{u}^* - \bar{u} > 0 \Leftrightarrow \frac{F_D}{F_U} > \bar{u} = \frac{\delta}{\delta+1}$.

# B Proof of Lemma 4.3

Assume that the current state $X_0 = x > h^*$ and consider the waiting time $\tau^*$ till the option to exit will be exercised. This is a random variable defined by $\tau^* = min\{t > 0 | X_t \leq h^*\}$. The expected waiting time (under the historic measure $\mathbb{P}$) can be calculated as follows:

$$\text{(A-1)} \qquad \begin{aligned} \mathbb{E}_{\mathbb{P}}[\tau_x^*] &= \mathbb{E}_{\mathbb{P}}^x\left[\int_0^\infty 1_{(h^*,\infty)}(\underline{X_t})dt\right] \\ &= \lim_{\gamma\to 0}\mathbb{E}_{\mathbb{P}}^x\left[\int_0^\infty e^{-\gamma t}1_{(h^*,\infty)}(\underline{X_t})dt\right] \\ &= \lim_{\gamma\to 0}\frac{1}{\gamma}\varepsilon^- 1_{(h^*,\infty)}(x). \end{aligned}$$

(A-1) uses an integral over time because in this paper the stochastic state variable $\{X_t\}_{t=0}^\infty$ follows a Brownian motion. And according to the probability distribution function of the Brownian motion the following could be calculated:

$$\text{(A-2)} \qquad \begin{aligned} \frac{1}{\gamma}\varepsilon^- 1_{(h^*,\infty)}(x) &= \frac{1}{\gamma}\int_{h^*-x}^0 (-\beta^-)e^{-\beta^- y}dy \\ &= \frac{1}{\gamma}\left[1 - e^{-\beta^{-}(h^*-x)}\right] = \frac{-\beta^-}{\gamma}(h^*-x). \end{aligned}$$

Please refer to chapter 10 in [1] for more details. Recall that from the characteristic equation, $\gamma - \mu\beta - \frac{\sigma^2}{2}\beta^2 = 0$:

$$\text{(A-3)} \qquad \beta^- = \frac{-\mu}{\sigma^2} - \frac{\sqrt{\mu^2 + 2\sigma^2\gamma}}{\sigma^2} = \frac{\gamma}{\mu},$$

where the last equality of (A-2) and (A-3) come from applying the Taylor's expansion. Finally, by plugging everything into (A-1), the analytical expression of expected waiting time is derived:

$$\text{(A-4)} \qquad \mathbb{E}_{\mathbb{P}}[\tau_x^*] = \lim_{\gamma\to 0}\frac{1}{\gamma}\frac{\gamma}{\mu}(h^*-x) = \frac{1}{\mu}(h^*-x) \in (0,\infty) \quad \forall x > h^*.$$

The expected waiting time is finite iff under the historic measure, the drift of the underlying factor $\mu$ is negative. And if $\mu < 0$, then the expected waiting time is inversely proportional to the drift. It is also proportional to the distance to the exit threshold and the current state.