Fertility, child-rearing cost and pay-as-you-go social security in a child-as-a-consumption-good model*

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Abstract

This paper asks the question; when does the expansion of the pay-as-you-go social security system decrease the fertility rate? The paper treats children as a consumption good instead of an investment good. Using an appropriately extended Diamond (1965)'s overlapping generations model, the paper shows the possibility of the decline of the fertility rate because of the increase of the pay-as-you-go social security system. The key for this result is how to treat the child-rearing cost.

1 Introduction

It is known that there is a strong negative correlation between fertility rate and the pay-as-you-go social security tax rate (see, for instance, Boldrin et al. (2005)). Not a few papers consider how lowering fertility rate affects the pay-as-you-go social security (see, for instance, Fanti and Gori (2011)). On the other hand, this paper investigates the opposite direction, i.e., how the expansion of the pay-as-you-go social security system affects the fertility rate.

There are several papers that ask the same question in different settings (see, for instance, Bental (1989), Wigger (1999), Boldrin et al. (2005), Hirazawa and Yakita (2009)). Among them, the closest paper to this paper is Boldrin et al. (2005). Boldrin et al. (2005) quantitatively analyzes the effect of the pay-as-you-go social security system to the fertility rate. This paper considers two types of well-known endogenous fertility models. One is a model based on Boldrin and Jones (2002), which assumes that parents procreate children because children support their parents when parents become old. The other is

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a model based on Becker and Barro (1988), which assumes that parents have children because parent's utility includes his/her children's utilities.[1] They conclude in their paper that the expansion of the pay-as-you-go social security system decreases the fertility rate in the first model significantly, while in the second model it does not significantly.

The difference between this paper and Boldrin et al. (2005) is how to treat children. In Boldrin et al. (2005), a child is regarded as an "investment" good. On the other hand, this paper regards children as a consumption good. As is written in Bental (1989), some think that the treatment of children as a consumption good is more appropriate for developed countries. Since developed countries are facing a drastic decline of the fertility rate, it is more meaningful to understand the mechanism of the declining of the fertility rate when children is treated as a consumption good.

The model of this paper is based on Diamond (1965), and the extended parts are that a young agent chooses how many children he/she has and a young agent pays the payroll tax to finance the social security payment. The main implication of this paper is that whether the pay-as-you-go social security system decreases the fertility rate or not depends on the type of the cost of child-rearing. In the study of the endogenous fertility how to treat the child-rearing cost is an issue. One possible cost of child-rearing is time, and the other is a physical good. This paper shows that the expansion of the pay-as-you-go social security system decreases the fertility rate only when the cost of child-rearing is a physical good. When time is a main source of the child-rearing cost, the fertility rate increases as the pay-as-you-go social security is expanding. The intuition for this result is the income effect and substitution effect due to the payroll tax. When the cost of child-rearing is time, we need to care about both substitution effect and income effect in order to study the effect of the expansion of the pay-as-you-go social security to the fertility rate, whereas only the income effect matters when the child-rearing cost is a physical cost.

The remaining of the paper is organized as follows. Section 2 investigates the case in which time is the cost of child-rearing. Section 3 considers the case in which physical goods are the cost of child-rearing. Section 4 concludes.

2 Time as the child-rearing cost

In this section, the child-rearing cost is time. That is, parents face trade off between working and having children.

2.1 Environment

Time is discrete and continues forever, $t = 1, 2, \ldots$. An agent lives for two periods: Young and old. A young agent is endowed with one unit of time. S/he distributes one unit of time between child-rearing and work. Let $\psi \in (0,1)$ be a time spending for one child. If a young agent at date t has n_t children, then s/he spends ψn_t of time for children and works for $1 - \psi n_t$. After production occurs, a young agent receives labor income, $(1 - \psi n_t)w_t$, and decides how much to consume, c_t^y and how much to save, s_t. A

[1] This is called a dynastic model.

young agent also has to pay payroll tax, which is denoted by $\tau \in [0,1)$. Thus, a young agent's budget constraint is

$$c_t^y + s_t = (1-\tau)(1-\psi n_t)w_t.$$

A young agent's saving at date t, s_t, is used for production at $t+1$ as a capital good without any cost of transforming from one unit of a consumption good to a capital good. When a young agent at date t saves s_t and s/he becomes old at $t+1$, s/he receives the interest income, $(1+R_{t+1})s_t$, where R_{t+1} is the net interest rate. Moreover, an old agent at date $t+1$ receives a social security, P_{t+1}. Since an old agent exists from the economy at the end of today, s/he has no incentive to save. Thus, an old agent at date $t+1$ faces the following budget constraint:

$$c_{t+1}^o = (1+R_{t+1})s_t + P_{t+1}.$$

An initial old agent has saved $s_0 > 0$ at date 0. The population of the initial old and the initial young are exogenously given, and they are $N_0 > 0$ and $N_1 > 0$ respectively.

An agent has a preference not only over consumption, but also on the number of children s/he has. A lifetime utility for an agent born at date t is represented by

$$U(c_t^y, n_t, c_{t+1}^o) := u(c_t^y) + \gamma v(n_t) + \beta u(c_{t+1}^o),$$

where $\beta \in (0,1]$ is a discount factor and $\gamma > 0$ is a preference weight relative to c_t^y. For analytical convenience, I assume $u(c) = \ln(c)$ and $v(n) = \ln(n)$.

At each date, a production occurs using labor and capital. Let L_t and K_t be an aggregate labor and capital. An aggregate labor at date t is $L_t = (1-\phi n_t)N_t$. Since capital at date t is supplied by young agents' saving, $K_t = N_{t-1}s_{t-1}$ holds. A production function is $F(K_t, L_t)$, and I assume $F(K,L) = K^\alpha L^{1-\alpha}$, where $\alpha \in (0,1)$. At each date t, a firm maximizes its profit,

$$F(K_t, L_t) - w_t L_t - r_t K_t,$$

where r_t is the rental rate of capital. I assume that factor markets are perfectly competitive. Assume further that a capital is fully depreciated once a production occurs. Hence, $r_t = 1 + R_t$. Notice that since $s_0 > 0$ and $N_0 > 0$ are given, $K_1 = N_0 s_0 > 0$ are given. Let k_t be a capital per labor at date t, i.e., $k_t = K_t/L_t$. Let $f(k) = k^\alpha$.

2.2 Allocation

Given K_1, N_0 and N_1, an *allocation* consists of a consumption sequence, $(c_1^o, (c_t^y, c_{t+1}^o)_{t=1}^\infty)$, a sequence of fertility, $(n_t)_{t=1}^\infty$ and a sequence of inputs for production, $(L_t, K_t)_{t=1}^\infty$. Given K_1, N_0 and N_1, an allocation is *feasible* if for all $t \geq 1$,

$$N_t c_t^y + N_{t-1} c_t^o + K_{t+1} = F(K_t, L_t),$$

$$L_t = (1-\psi n_t)N_t,$$

and

$$N_{t+1} = n_t N_t.$$

2.3 Government

At each period, a government collects payroll tax from young agents and pays a social security to old agents. I assume this social security scheme must be budget balanced at each date, that is,

$$N_t P_{t+1} = N_{t+1}\tau(1-\psi n_{t+1})w_{t+1}$$

holds for all $t \geq 0$.

2.4 Equilibrium

An equilibrium concept is a perfect foresight competitive equilibrium. Here is a formal definition.

Definition 2.1. *Given $K_1 > 0$, $N_0 > 0$, $N_1 > 0$ and $\tau \in [0,1)$, a* **competitive equilibrium** *consists of a feasible allocation, a sequence of savings, $(s_t)_{t=1}^{\infty}$, a sequence of prices $(w_t, r_t)_{t=1}^{\infty}$ and a sequence of pensions, $(P_t)_{t=1}^{\infty}$, such that*

1. *Given prices and pensions, a young agent solves*

$$\begin{array}{cl} \max_{c_t^y, s_t, n_t, c_{t+1}^o} & \ln(c_t^y) + \gamma \ln(n_t) + \beta \ln(c_{t+1}^o) \\ s.t. & c_t^y + s_t = (1-\tau)(1-\psi n_t)w_t \\ & c_{t+1}^o = r_{t+1}s_t + P_{t+1} \end{array}$$

2. *For the initial old, $c_1^o = r_1 \frac{K_1}{N_0} + P_1$.*

3. *THe firm maximizes its profit at each date:*

$$w_t = (1-\alpha)k_t^{\alpha},\ r_t = \alpha k_t^{\alpha-1}$$

4. *A capital market and labor market clear: $K_t = N_{t-1}s_{t-1}$ and $L_t = N_t(1-\psi n_t)$.*

5. *The government budget constraint is satisfied, i.e., $N_{t-1}P_t = N_t\tau(1-\psi n_t)w_t$.*

Since a sequence of capital per capita and fertility rate characterizes a competitive equilibrium, I call a sequence, $(k_t, n_t)_{t=1}^{\infty}$, a competitive equilibrium from now on. A competitive equilibrium, (k_t, n_t), *converges to a steady state*, (k, n), if $(k_t, n_t) \to (k, n)$ as t goes to infinity. I call such (k, n) a *steady state*.

2.5 Existence of a unique competitive equilibrium and a unique steady state

The first-order conditions for an interior solution to an agent's problem are

$$s_t: \quad -\frac{1}{(1-\tau)(1-\psi n_t)w_t - s_t} + \frac{\beta r_{t+1}}{r_{t+1}s_t + P_{t+1}} = 0 \tag{1}$$

$$n_t: \quad -\frac{(1-\tau)w_t\psi}{(1-\tau)(1-\psi n_t)w_t - s_t} + \frac{\gamma}{n_t} = 0. \tag{2}$$

From (1) and (2), we have

$$s_t = \frac{\beta(1-\tau)}{1+\beta+\gamma}w_t - \frac{(1+\gamma)P_{t+1}}{(1+\beta+\gamma)r_{t+1}}, \tag{3}$$

and

$$n_t = \frac{\gamma}{\psi(1+\beta+\gamma)} + \frac{\gamma P_{t+1}}{(1-\tau)\psi(1+\beta+\gamma)w_t r_{t+1}} \tag{4}$$

Using (3) and $P_{t+1} = n_t\tau(1-\psi n_{t+1})w_{t+1}$, capital per labor is

$$k_{t+1} = \frac{s_t}{(1-\psi n_{t+1})n_t} = \frac{\beta(1-\tau)}{1+\beta+\gamma}\frac{w_t}{(1-\psi n_{t+1})n_t} - \frac{(1+\gamma)\tau w_{t+1}}{(1+\beta+\gamma)r_{t+1}}. \tag{5}$$

In an equilibrium, $w_t = (1-\alpha)k_t^\alpha$, $r_t = \alpha k_t^{\alpha-1}$ and $P_{t+1} = n_t\tau(1-\psi n_{t+1})w_{t+1}$ hold. Plugging them into (5) and (4), the law of motions of capital per labor and fertility are derived:

$$k_{t+1} = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)}\frac{k_t^\alpha}{(1-\psi n_{t+1})n_t} \tag{6}$$

and

$$n_t = \frac{(1-\tau)\alpha\gamma k_t^\alpha}{(1-\tau)\psi(1+\beta+\gamma)\alpha k_t^\alpha - \gamma(1-\psi n_{t+1})\tau k_{t+1}}. \tag{7}$$

Letting $z_{t+1} := \frac{k_{t+1}}{k_t^\alpha}$, (6) and (7) become

$$z_{t+1} = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)}\frac{1}{(1-\psi n_{t+1})n_t} \tag{8}$$

and

$$n_t = \frac{(1-\tau)\alpha\gamma}{(1-\tau)\psi(1+\beta+\gamma)\alpha - \gamma(1-\psi n_{t+1})\tau z_{t+1}}. \tag{9}$$

Plugging (8) into (9) with arrangement, we have

$$n_t = \frac{\gamma}{\psi} \frac{\alpha + \tau(1-\alpha)}{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)} > 0. \tag{10}$$

Let

$$\overline{n} := \frac{\gamma}{\psi} \frac{\alpha + \tau(1-\alpha)}{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)}. \tag{11}$$

Note that the fertility converges to $\overline{n}$ after one period.

Plugging (10) into (8), we have

$$k_{t+1} = \frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)} \frac{1}{(1-\psi\overline{n})\overline{n}} k_t^{\alpha}. \tag{12}$$

Proposition 2.1. *There exists a unique competitive equilibrium, and its allocation is characterized by (10) and (12).*

(12) guarantees a unique steady state.

Proposition 2.2. *A competitive equilibrium converges to a unique steady state, and the fertility rate in a steady state is (11) and capital stock is*

$$\overline{k} = \left[\frac{\alpha\beta(1-\tau)(1-\alpha)}{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)} \frac{1}{(1-\psi\overline{n})\overline{n}} \right]^{\frac{1}{1-\alpha}}. \tag{13}$$

Here is the main result of this section.

Proposition 2.3. *In a steady state, the fertility rate increases as the payroll tax rate increases, i.e., $\overline{n}$ is strictly increasing in τ.*

Proof. Let

$$B := (1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha).$$

Taking the derivative of $\overline{n}$ with respect to τ,

$$\begin{aligned} \frac{d\overline{n}}{d\tau} &= \frac{\gamma}{\psi}\frac{1}{B^2}\left[(1-\alpha)\left\{(1+\beta+\gamma)\alpha + (1+\gamma)\tau(1-\alpha)\right\} - \left\{\alpha + \tau(1-\alpha)\right\}(1+\gamma)(1-\alpha)\right] \\ &= \frac{\gamma}{\psi}\frac{1}{B^2}\alpha\beta(1-\alpha) > 0. \end{aligned}$$

This completes the proof. *Q.E.D.*

3 A physical good as the child-rearing cost

Next, consider a case in which child-rearing cost is a physical good. In words, parents have to feed their children to raise.

3.1 Environment

Time is discrete and continues forever, $t = 1, 2, \ldots$. An agent lives for two periods: Young and old. When an agent is young, s/he provides one unit of labor inelastically. After production occurs, a young agent receives labor income, w_t, and decides how much to consume, c_t^y how much to save, s_t, and how many children s/he has, n_t. To raise one child, a young agent spends ϕ_t per child. In addition, a young agent needs to pay payroll tax, which is denoted by $\tau \in [0, 1)$. Thus, a young agent's budget constraint is

$$c_t^y + s_t + \phi_t n_t = (1 - \tau) w_t. \tag{14}$$

A young agent's saving at date t, s_t, is used for production at $t+1$ as a capital good without any cost of transforming from one unit of a consumption good to a capital good. When a young agent at date t saves s_t and s/he becomes old at $t+1$, s/he receives the interest income, $(1 + R_{t+1}) s_t$, where R_{t+1} is net rate of return to savings. Moreover, an old agent at date $t+1$ receives a pension, P_{t+1}. Since an old agent exits from the economy at the end of today, s/he has no incentive to save. Thus, an old agent at date $t+1$ faces the following budget constraint:

$$c_{t+1}^o = (1 + R_{t+1}) s_t + P_{t+1}.$$

An initial old agent has saved $s_0 > 0$ at date 0. The population of the initial old and the initial young are exogenously given, and they are $N_0 > 0$ and $N_1 > 0$ respectively.

An agent has a preference not only over consumption, but also on the number of children s/he has. For simplicity, I assume that consumption when child does not affect lifetime utility. Thus, a lifetime utility for an agent born at date t is represented by

$$U(c_t^y, n_t, c_{t+1}^o) := u(c_t^y) + \gamma v(n_t) + \beta u(c_{t+1}^o),$$

where $\beta \in (0, 1]$ is a discount factor and $\gamma > 0$ is a relative preference weight to c_t^y. For analytical convenience, I assume $u(c) = \ln(c)$ and $v(n) = \ln(n)$.

At each date, a production occurs using labor and capital. Let L_t and K_t be an aggregate labor and capital. Since labor is inelastically supplied by young agents, $L_t = N_t$ holds at any date t. Moreover, since capital is supplied by young agents' saving, $K_t = N_{t-1} s_{t-1}$ holds. A production function is $F(K_t, L_t)$, and I assume $F(K, L) = K^\alpha L^{1-\alpha}$, where $\alpha \in (0, 1)$. At each date t, a firm maximizes its profit,

$$F(K_t, L_t) - w_t L_t - r_t K_t.$$

I assume that factor markets are perfectly competitive. Assume further that a capital is fully depreciated once a production occurs. Therefore, $1 + R_t = r_t$ holds. Notice that since $s_0 > 0$ and $N_0 > 0$ are given, $K_1 = N_0 s_0 > 0$ are given. Let k_t be a capital per labor at date t, i.e., $k_t = K_t / L_t$. Let $f(k) = k^\alpha$.

3.2 Allocation

Given K_1, N_0 and N_1, an *allocation* consists of a consumption sequence, $(c_1^o, (c_t^y, c_{t+1}^o)_{t=1}^{\infty})$, a sequence of fertility, $(n_t)_{t=1}^{\infty}$ and a sequence of inputs for production, $(L_t, K_t)_{t=1}^{\infty}$. Given K_1, N_0 and N_1, an allocation is *feasible* if for all $t \geq 1$,

$$N_{t+1}\phi_t + N_t c_t^y + N_{t-1} c_t^o + K_{t+1} = F(K_t, L_t),$$

$$L_t = N_t,$$

and

$$N_{t+1} = n_t N_t.$$

3.3 Government

At each period, a government collects payroll tax from young agents and pays a pension to old agents. I assume this pension scheme must be budget balanced at each date, that is,

$$N_t P_{t+1} = N_{t+1} \tau w_{t+1}$$

holds for all $t \geq 0$.

3.4 Equilibrium

An equilibrium concept is a perfect foresight competitive equilibrium. Here is a formal definition.

Definition 3.1. *Given $K_1 > 0$, $N_0 > 0$, $N_1 > 0$ and $\tau \in [0,1)$, a* **competitive equilibrium** *consists of a feasible allocation, a sequence of savings, $(s_t)_{t=1}^{\infty}$, a sequence of prices $(w_t, r_t)_{t=1}^{\infty}$ and a sequence of pensions, $(P_t)_{t=1}^{\infty}$, such that*

1. *Given prices and pensions, a young agent solves*

$$\begin{array}{cl} \max_{c_t^y, s_t, n_t, c_{t+1}^o} & \ln(c_t^y) + \gamma \ln(n_t) + \beta \ln(c_{t+1}^o) \\ s.t. & c_t^y + s_t + \phi_t n_t = (1-\tau) w_t \\ & c_{t+1}^o = r_{t+1} s_t + P_{t+1} \end{array}$$

2. *For the initial old, $c_1^o = r_1 \frac{K_1}{N_0} + P_1$.*

3. *The firm maximizes its profit:*

$$w_t = (1-\alpha) k_t^{\alpha},\ r_t = \alpha k_t^{\alpha - 1}$$

4. *A capital market clears:* $K_t = N_{t-1}s_{t-1}$.

5. *The government budget constraint is satisfied, i.e.,* $N_{t-1}P_t = N_t \tau w_t$.

As we will see, a competitive equilibrium is characterized by a sequence of k_t and n_t. I call a pair of sequence $(k_t, n_t)_{t=1}^{\infty}$ a *competitive equilibrium* if an allocation derived from $(k_t, n_t)_{t=1}^{\infty}$ constitutes a competitive equilibrium. A competitive equilibrium *converges to a steady state*, (k,n), if a competitive equilibrium (k_t, n_t) converges to (k,n) as t goes to infinity. I call a (k,n) where a competitive equilibrium converges a *steady state*.

For instructive reason, I impose one assumption to ϕ_t.

Assumption 3.1. *For* $\psi \in (0,1)$, $\phi_t := \psi w_t$.

This form is also used in Wigger (1999), Boldrin and Jones (2002) and Fanti and Gori (2011). The interpretation of this assumption is that if a parent is rich, then a parent spends more money for his/her kid, and vice versa.[2]

3.5 Existence of a unique competitive equilibrium

First, a competitive equilibrium is analyzed.

The first-order conditions for an interior solution to an agent's problem are

$$s_t: \quad -\frac{1}{w_t(1-\tau) - \psi w_t n_t - s_t} + \frac{\beta r_{t+1}}{r_{t+1}s_t + P_{t+1}} = 0 \tag{15}$$

$$n_t: \quad -\frac{\psi w_t}{w_t(1-\tau) - \psi w_t n_t - s_t} + \frac{\gamma}{n_t} = 0. \tag{16}$$

From (15) and (16), we have

$$s_t = \frac{\beta(1-\tau)}{1+\beta+\gamma} w_t - \frac{1+\gamma}{(1+\beta+\gamma)r_{t+1}} P_{t+1}, \tag{17}$$

and

$$n_t = \frac{\gamma(1-\tau)}{(1+\beta+\gamma)\psi} + \frac{\gamma}{\psi w_t r_{t+1}(1+\beta+\gamma)} P_{t+1} \tag{18}$$

Plugging the government budget constraint, $P_{t+1} = n_t w_{t+1}\tau$, into (18) and rearranging terms, we have

$$n_t = \frac{w_t r_{t+1}(1-\tau)\gamma}{\psi w_t r_{t+1}(1+\beta+\gamma) - \gamma w_{t+1}\tau}. \tag{19}$$

Since $K_{t+1} = N_t s_t$, $k_{t+1} = \frac{s_t}{n_t}$. Using (17), (19) and equilibrium price conditions, we have

$$k_{t+1} = \frac{\alpha\beta\psi(1-\alpha)}{[\alpha + (1-\alpha)\tau]\gamma} k_t^{\alpha} \tag{20}$$

[2]Even if $\phi_t = \phi \in (0,1)$, we get the same implication. For more details, see the Appendix.

Proposition 3.1. *Given $\tau \in [0,1)$, $K_1 > 0$, $N_0 > 0$ and $N_1 > 0$, there is a unique competitive equilibrium and it is characterized by (19) and (20).*

From (20), k_t converges to a unique point. This implies n_t defined by (19) also converges to some constant n.

Proposition 3.2. *A competitive equilibrium converges to a unique $(\bar{k},\bar{n})$, where*

$$\bar{k} = \left[\frac{\alpha\beta\psi(1-\alpha)}{\{\alpha+(1-\alpha)\tau\}\gamma}\right]^{\frac{1}{1-\alpha}} \tag{21}$$

and

$$\bar{n} = \frac{\gamma(1-\tau)}{\psi}\frac{\alpha+(1-\alpha)\tau}{(1+\beta+\gamma)\alpha+(1+\gamma)\tau(1-\alpha)}. \tag{22}$$

3.6 Characterization of a steady state

First result is the characterization of a steady-state capital per capita.

Proposition 3.3. *A steady-state capital per capita is strictly decreasing in τ.*

Proof. In (21), τ appears only in the denominator. Therefore, it is strictly decreasing in τ. *Q.E.D.*

Taking the derivative of a steady-state $\bar{n}$ with respect to τ,

$$\frac{\partial\bar{n}}{\partial\tau} = \frac{\psi\gamma\{(1-\alpha)(1-\tau)\alpha\beta - [\alpha+(1-\alpha)\tau]\alpha(1+\beta+\gamma) - [\alpha+(1-\alpha)\tau](1-\alpha)\tau(1+\gamma)\}}{D}, \tag{23}$$

where $D := [\psi\{\alpha(1+\beta+\gamma)+(1-\alpha)\tau(1+\gamma)\}]^2 > 0$. The sign of $\frac{\partial\bar{n}}{\partial\tau}$ is the same as that of the numerator of (23). Rearranging the numerator of (23), the sign is determined by

$$\begin{aligned} F(\tau) \quad := \quad & -(1-\alpha)^2(1+\gamma)\tau^2 - \{(1-\alpha)\alpha\beta+(1-\alpha)\alpha(1+\beta+\gamma)+\alpha(1-\alpha)(1+\gamma)\}\tau \\ & +(1-\alpha)\alpha\beta-\alpha^2(1+\beta+\gamma). \end{aligned}$$

Notice that F is strictly decreasing in τ on $[0,1)$. At $\tau = 1$,

$$F(1) = -(1-\alpha)^2(1+\gamma)-(1-\alpha)\alpha(1+\beta+\gamma)-\alpha(1-\alpha)(1+\gamma)-\alpha^2(1+\beta+\gamma) < 0.$$

At $\tau = 0$,

$$F(0) = \alpha\{(1-\alpha)\beta-\alpha(1+\beta+\gamma)\}.$$

Since F is continuous and strictly decreasing in τ on $[0,1)$, the sign of $\frac{\partial\bar{n}}{\partial\tau}$ depends on the sign of $F(0)$.

Proposition 3.4. *If $0 < \alpha < \frac{\beta}{1+\gamma+2\beta}$, then the fertility rate in a steady state, $\bar{n}$, is inverse U-shaped with respect to τ. If $\frac{\beta}{1+\gamma+2\beta} < \alpha < 1$, then the fertility rate in a steady state is strictly decreasing in τ.*

Let us consider the condition,

$$\alpha < \frac{\beta}{1+\gamma+2\beta}. \tag{24}$$

Suppose $\tau = 0$, and consider a steady-state interest rate, $\bar{r}$, and fertility rate, $\bar{n}$. By (21),

$$\bar{k}(0) = \left[\frac{\beta\psi(1-\alpha)}{\gamma}\right]^{\frac{1}{1-\alpha}}.$$

From this,

$$\bar{r}(0) = \frac{\alpha\gamma}{\beta\psi(1-\alpha)}.$$

By (22),

$$\bar{n}(0) = \frac{\gamma}{\psi(1+\beta+\gamma)}.$$

From these $\bar{r}(0)$ and $\bar{n}(0)$,

$$\bar{n}(0) > \bar{r}(0) \Leftrightarrow (24).$$

If (24) holds, in a stationary equilibrium, the fertility rate is greater than the interest rate as long as the payroll tax is small enough because of the continuity of r and n in τ. Since n is the return from pay-as-you-go pension and r is the return from the saving, a young agent invests more on the "asset" whose rate of return is higher. Note that an increase of pay-as-you-go pension decreases the saving. In addition, since the fertility rate increases, per capita capital decreases, and the interest rate falls. This continues until the rate of return from pay-as-you-go pension and that from saving are equalized.

4 Conclusion

To understand the mechanism that gives the difference between two models, let us look at young agent's budget constraints. When time is the child-rearing cost, it is

$$c_t^y + s_t + (1-\tau)\psi w_t n_t = (1-\tau)w_t$$

and when a physical good is the cost of child-rearing, it is

$$c_t^y + s_t + \psi w_t n_t = (1-\tau)w_t.$$

As we see, when τ increases, the labor income also decreases in both models. However, when time is the cost of child-rearing, an increase in τ decreases the opportunity of child-rearing cost. This stimulates the incentive to have children. On the other hand, when a physical good is the cost of child-rearing, there is no such effect. Therefore, when a physical good is the child-rearing cost, an increase in the payroll tax rate tends to decrease the fertility rate.

A Constant physical good as the child-rearing cost

In section 3, I impose one assumption on the form of the child-rearing cost. In this appendix, I show that even when the child-rearing cost is ψn_t, the same result as Proposition 3.4 holds.

Only difference is a young agent's budget constraint. (14) changes into

$$c_t^y + s_t + \psi n_t = (1-\tau)w_t.$$

Following the same procedure to find a competitive equilibrium, we will have

$$k_{t+1} = \frac{\alpha}{\alpha+(1-\alpha)\tau}\frac{\beta\psi}{\gamma} \tag{25}$$

and

$$n_t = \frac{\alpha k_{t+1}^{\alpha-1}\gamma(1-\tau)(1-\alpha)k_t^{\alpha}}{\psi(1+\beta+\gamma)\alpha k_{t+1}^{\alpha-1} - \gamma\tau(1-\alpha)k_{t+1}^{\alpha}}. \tag{26}$$

A unique competitive equilibrium is characterized by (25) and (26). Capital per labor is converging in one period in this case.

Plugging (25) into (26), we will find a steady-state $(\bar{k}, \bar{n})$ such that

$$\bar{k} = \frac{\alpha}{\alpha+(1-\alpha)\tau}\frac{\beta\psi}{\gamma}$$

and

$$\bar{n} = \frac{\gamma(1-\alpha)\left(\frac{\alpha\beta\psi}{\gamma}\right)^{\alpha}(1-\tau)\left\{\alpha+(1-\alpha)\tau\right\}^{1-\alpha}}{\psi(1+\beta+\gamma)\alpha+\psi(1-\alpha)\tau(1+\gamma)}$$

Taking the derivative of $\bar{n}$ with respect to τ, we have

$$\begin{aligned}\frac{d\bar{n}}{d\tau} &= \frac{\gamma(1-\alpha)\left(\frac{\alpha\beta\psi}{\gamma}\right)^{\alpha}}{D}\left\{-[\alpha+(1-\alpha)\tau]^{1-\alpha}+(1-\tau)(1-\alpha)^2[\alpha+(1-\alpha)\tau]^{-\alpha}\right\} \\ &- \frac{\gamma(1-\alpha)\left(\frac{\alpha\beta\psi}{\gamma}\right)^{\alpha}(1-\tau)[\alpha+(1-\alpha)\tau]^{1-\alpha}}{D^2}\psi(1-\alpha)(1+\gamma),\end{aligned}$$

where $D := \psi(1+\beta+\gamma)\alpha+\psi(1-\alpha)\tau(1+\gamma)$. The sign of $\frac{d\bar{n}}{d\tau}$ is the same as the sign of

$$\begin{aligned}&[(1+\beta+\gamma)\alpha+(1-\alpha)\tau(1+\alpha)]\left[-\{\alpha+(1-\alpha)\tau\}+(1-\tau)(1-\alpha)^2\right] \\ -\ &(1-\tau)[\alpha+(1-\alpha)\tau](1-\alpha)(1+\gamma),\end{aligned}$$

which is the numerator of $\frac{d\bar{n}}{d\tau}$ after some rearrangements. From this, let

$$\begin{aligned} H(\tau) \quad := \quad & -(1+\gamma)(1-\alpha)^3\tau^2 + \left\{-(1+\beta+\gamma)\alpha(1-\alpha)(2-\alpha) - (1+\gamma)(1-\alpha)^2\alpha - (1-\alpha)^2(1+\gamma)\right\}\tau \\ & +(1+\beta+\gamma)\alpha[-\alpha+(1-\alpha)^2] - (1-\alpha)(1+\gamma)\alpha. \end{aligned}$$

It is not difficult to check H is strictly decreasing in τ on $(0,1)$, and $H(1) < 0$. Therefore, the sign of $\frac{d\bar{n}}{d\tau}$ depends on the sign of $H(0)$. When $\tau = 0$,

$$\begin{aligned} H(0) &= -(1+\beta+\gamma)\alpha^2 + (1+\beta+\gamma)\alpha(1-\alpha)^2 - (1-\alpha)(1+\gamma)\alpha \\ &= \alpha\left[(1+\beta+\gamma)\alpha^2 - (2+3\beta+2\gamma)\alpha + \beta\right]. \end{aligned}$$

Let

$$h(\alpha) := (1+\beta+\gamma)\alpha^2 - (2+3\beta+2\gamma)\alpha + \beta.$$

Then, $h(1) = -1-\beta-\gamma < 0$ and $h(0) = \beta > 0$. Again, h is strictly decreasing in α on (0,1). Since h is continuous in α, this implies that there exists a unique $\widehat{\alpha} \in (0,1)$ such that for $\alpha < \widehat{\alpha}$, $h(\alpha) > 0$ and for $\alpha > \widehat{\alpha}$, $h(\alpha) < 0$. Hence, when $\alpha < \widehat{\alpha}$, $H(0) > 0$ and when $\alpha > \widehat{\alpha}$, $H(0) < 0$. Therefore:

Proposition A.1. *There exists a unique $\widehat{\alpha} \in (0,1)$ such that (i) if $\alpha < \widehat{\alpha}$, $\bar{n}$ is inverse U-shaped in τ; (ii) if $\alpha > \widehat{\alpha}$, $\bar{n}$ is strictly decreasing in τ.*

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