

**Shark Attack:**  
**Reasonable Royalty and the Division of Profit for**  
**Probabilistic Patents\***

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This Version: March, 2012

**Abstract**

We analyze the incentive to innovate for a patent holder under probabilistic patents. In a sequential innovation framework, actual patent infringement occurs and there exists uncertainty in litigation with the damage award adjudicated as a reasonable royalty. On the one hand, complete profit transfer from the second generation innovator proves feasible for medium degrees of the second generation innovation. On the other hand, both minuscule and significant innovations render infringement suits unprofitable for the patent holder. Hence, the patent holder refrains from patent litigation and no profit transfer occurs. We further discuss the effects of varying patent breadth, either by an adjustment in the written law or by application of the doctrine of equivalents.

*JEL Classification:* O31; O34; K41.

*Keywords:* Probabilistic patents, reasonable royalty, sequential innovation, doctrine of equivalents.

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\* This paper was formerly circulated under the title “Reasonable Royalty and the Division of Profit in Sequential Innovation.” We thank Salvatore Capasso for thoughtful comments.

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## I. INTRODUCTION

The division of profit between sequential innovators who are caught in the shadow of patent litigation has recently drawn a great deal of attention from economists, innovators and patent law practitioners. What lies at the heart of the debates and discussions is that patents are not well defined property right and hence are probabilistic (Lemley and Shapiro, 2005). Patents are probabilistic in their research and development phase because of the very intrinsic nature of innovation; patents are probabilistic in the ensuing application and grant phase of information asymmetry due to the functionality of Patent Office; finally, patents are probabilistic in the phase of enforcement. A recent litigation vividly illustrates the highly complicated nature and the consequently intertwined effects of a probabilistic patent. NTP Co., a textbook example of a “patent shark”<sup>1</sup> who had acquired wireless email patents sued Research in Motion (RIM) for its Blackberry devices and services. Being threatened to shut down its US operation, RIM paid NTP more than a half billion dollars.

Despite the fact that economists have been working on issues of intellectual property rights for a long time, the approach taken by the “patent race” literature so far cannot identify all the complex effects of patent protection when innovation is cumulative and the ensuing intellectual property rights are probabilistic. An important research agenda aiming for a critical assessment of patents has been pursued by Green and Scotchmer (1995). Their work examines the issue of optimal patent breadth and duration, and the role of different legal mechanisms for cumulative innovations. What emerges from their analysis as the major patent policy concern is the lack of incentive to innovate for the potential patent holder in the shadow of incomplete transfer of social value facilitated by basic research. Hence longer-lasting patents are called for as the remedy for insufficient innovation incentive.

Green and Scotchmer (1995) confine themselves to the “fencepost” interpretation of the patent system upon which the majority of the patent literature has been built. For example, Hortsmann et al (1995) study the incentives provided by a “limited but exact coverage” patent system to become a successful innovator (the sole winner of a patent race). Almost by definition, important policy implications regarding the litigation process cannot be addressed within a perfect fencepost system. In reality, patent breadth is *de facto* a matter of Patent Office and court interpretation.

This article aims to fill the void left by the fencepost patent literature and explore

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<sup>1</sup> According to Henkel and Reitzig (2008), patent sharks are “firms with hidden intellectual property that surface, threatening to sue, when their rights are inadvertently infringed.”

the following two important aspects of probabilistic patents: uncertainty in patent litigation and asymmetric bargaining power.

**Uncertainty in Patent Litigation.** Lemley and Shapiro (2005) recently emphasize that “uncertainty about validity and scope are critical when studying the enforcement and litigation of patents.” Our “signpost” interpretation of the patent system takes into account the fact that patent protection is imperfect, that the patent system often cannot circumscribe its objects, individual patents, in a precise and unquestionable way – in the phases of granting and enforcing patents. Accordingly, a patent is described by Kitti (1978) as a “lottery ticket”.

In an earlier attempt to capture this aspect of reality, Waterson (1990) looked at uncertainty in patent infringement litigation and employed the concept of “limited but inexact patent coverage” in a horizontal product differentiation model. Pursuing this line of research one step further in a vertical product differentiation model, Chou and Haller (2007) find that when the inherent uncertainty of a patent infringement case is taken into consideration, the division of profit between sequential innovators will depend on the degree of the improvement made by the subsequent innovator. In particular, within a wide range of model parameters, full rent extraction by the initial patent holder is possible.

Uncertainty in litigation is a common feature of our papers and the litigation literature. Also, the litigation process assumed here and in Chou and Haller (2007) is similar to the one most frequently used in the litigation literature. Among the precedents are Reinganum and Wilde (1986), Meurer (1989), and Aoki and Hu (1995). In their models, the plaintiff (the harmed party) makes a settlement offer to the defendant (the party who inflicts harm). The defendant then responds by either accepting the offer (take-it) or refusing the settlement proposal (leave-it), after which a court action may be taken. The most crucial modeling differences between Chou and Haller (2007) and the present paper on the one side and the aforementioned litigation literature on the other side are twofold. First of all, the litigation literature focuses on asymmetric information between patentees and infringers about costs (harm, damages) or benefits (surplus, profits).<sup>2</sup> In the presence of asymmetric information, sequential equilibrium is the predominant solution concept. In contrast, our model assumes symmetric information. Hence we adopt subgame perfect equilibrium as the appropriate solution concept. Secondly, the conventional litigation literature works with a given invention and a fixed probability that the patentee wins a patent infringement suit whereas in our approach, the degree of improvement derived from a basic innovation determines the patentee’s probability of winning infringement

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<sup>2</sup> See, e.g., Bebchuck (1984), Reinganum and Wilde (1986), and Meurer (1989).

litigation.

**Asymmetric Bargaining Power.** Green and Scotchmer (1995) assume that any agreement between the two parties means an equal split of the available surplus. Nevertheless anecdotal evidence and the more frequent use of preliminary injunction point to a paradigmatic shift in favor of patent holders, giving them substantial bargaining power. For instance, the Kodak-Polaroid dispute has been touted as the “most prominent example of an increasingly pro-patent sentiment in American courts”.<sup>3</sup> In terms of procedural and doctrinal changes, the new attitude manifests itself through greater leniency of granting preliminary injunctions. Since a credible threat is at work in halting the subsequent innovator’s entire operation by means of the injunctive measure, a scenario in which patent holder makes the first move with a take-it-or-leave-it licensing offer seems worth exploring.

**Royalties and Damage.** The model of the present paper differs from Chou and Haller (2007) in an unobtrusive yet important detail: how profit is divided, if the patent holder prevails in court. In Chou and Haller (2007), after rendering the verdict of infringement, the court always grants the patent holder what she has demanded as a licensing fee at the very beginning of bargaining. By and large, the analysis of that model remains valid under the additional stipulation that the court-imposed royalty must not exceed the total profit. Note that prior to 1946 a patent holder was allowed to choose between the amount of damages she suffered and the amount of profit earned by the infringer. Since then, it is the courts rather than the plaintiffs who determine royalties and damages paid by infringers.

If determination of royalties and damages is left to the courts’ discretion, then the issue of reasonable royalty arises and ought to be addressed. Since 1946, several doctrines of reasonable royalty have been applied.<sup>4</sup> In the current model, we postulate that after a verdict of infringement the court awards a royalty equal to the profit generated by the infringer’s commercial sales. Coupled with the other specifications of our model, this assumption proves consistent with the prevailing doctrines. One might think that if the court is capable of assessing the right amount of reasonable royalty, the assumption of uncertain or unsound judgment associated with the legal system seems less justifiable. Notice, however, that both in our model and in practice the sequencing in infringement suits is such that damages are awarded after a finding of infringement. Therefore, we consider the two functions of the court – verifying the validity of a patent and ruling on infringement versus awarding the damages – as two distinct events.<sup>5</sup> Whereas the model assumes uncertainty about who wins an

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<sup>3</sup> Schmitt, E. “Business and the Law: Judicial Shift in Patent Cases,” *New York Times*, Jan. 21, 1986.

<sup>4</sup> We provide a brief survey in appendix A.

<sup>5</sup> Typically, these functions are even exercised by two different courts, the CAFC and trial courts, respectively.

infringement suit, it does not postulate stochastic damages in case the court finds the defendant guilty of infringement: the court, according to the *ex post* factual findings, always grants the consistent and accurate reasonable royalty.<sup>6</sup>

**Doctrine of Equivalents.** It has long been recognized that patent breadth is a viable instrument to influence inventors' incentives to innovate. Though not explicitly stated, theorists interested in the design of an optimal patent system seem to suggest that the optimal patent breadth should be established through an adjustment in the written law (Klemperer, 1990; Gilbert and Shapiro, 1990; Green and Scotchmer, 1995). However, patent protection can also be expanded (or conversely, contracted) over time through flexible application of legal doctrines without resorting to specific changes in the Patent Codes. In particular, the applications of the doctrine of equivalents and the reverse doctrine of equivalents constitute important instances of such flexibility. The doctrine of equivalents applies to the patent representing a "pioneer invention" – which the US Supreme Court has defined as "a patent concerning a function never before performed, a wholly novel device, or one of such novelty and importance as to make a distinct step in the progress in the art," [*Boyden Power-Brake Co. v. Westinghouse*, 170 U.S. 537, 569 (1898)]. As a symmetric leverage to the doctrine of equivalents, the reverse doctrine of equivalents "is an equitable doctrine invoked in applying properly construed claims to an accused device. Just as the purpose of the 'doctrine of equivalents' is to prevent 'pirating', the purpose of the 'reverse doctrine of equivalents' is to prevent unwarranted extension of the claim beyond a fair scope of the patentee's invention" [*Scripps Clinic & Research Fund. v. Genetech, Inc.*, 927 F.2d 1565, 18 U.S.P.Q.2d (BNA) 1001, 18 U.S.P.Q.2d (BNA) 1896 (Fed. Cir. 1991)].

**Summary of Results.** Three main results in Chou and Haller (2007) are still obtained under the new specification of reasonable royalty. That is, (1) a complete profit transfer equilibrium is attainable under a wide range of model parameters, (2) higher patent infringement litigation cost may dampen the patent holder's incentive to innovate, and (3) a broader patent breadth may not unreservedly improve the patent holder's incentive to innovate.

One major limitation of Chou and Haller (2007) is that technical intractability confines the analysis only to the complete-profit-transfer equilibrium hence the richness of the strategic interaction within other type of equilibria is left unexplored. In the current framework we are able to partition the statutory patent breadth into three areas that can be characterized by three types of equilibria: i.e., complete-profit-transfer Take-it equilibrium, incomplete-profit-transfer Take-it

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<sup>6</sup> Our theoretic underpinning of the reasonable royalty is also consistent with Hypothesis 1 (*circularity of damage doctrines and licensing offer*) discussed in Schankerman and Scotchmer (2001:205).

equilibrium, and No-Action equilibrium, respectively. The strength (effectiveness) of the patent system can thus be more comprehensively measured. The range of innovation that corresponds to the No-Action equilibrium provides the strongest incentive to the infringing firm to “trespass” the intellectual property of the patent holder. This area consists of a low-end interval adjacent to the basic innovation and another high-end interval adjacent to the upper boundary point of the patent breadth. In other words, “inventing around” (Gallini, 1992) with close imitation or “inventing enough” with quite a novel (though still infringing) product may be observed in our model without seeing the patent holder taking legal action.<sup>7</sup>

In addition to the length of the complete-profit-transfer interval, the new framework allows for computing alternative measures of the efficacy of the patent system in transferring profits: the average expected payoff for the patent holder, both in absolute terms and as percentage of average expected profit.

The paper is organized as follows. In Section 2 we set forth the model to be used to investigate the division of profit between sequential innovations. In Section 3 we perform equilibrium analysis. Section 4 presents the comparative statics with respect to infringement litigation costs. Moreover, we look at the instruments used by the courts to broaden (shorten) the patent breadth: the doctrine of equivalents and the reverse doctrine of equivalents. Some concluding remarks are offered in Section 5. Appendix A elaborates on doctrines of reasonable royalty. Appendix B contains more technical derivations of results.

## II . The Model

We begin with the development of the basic model. There are one research institution and one firm. The research institution has acquired a patent on its invention of quality  $x$  and is denoted as the patent holder (PH) hereafter. We set  $x = 0$  without loss of generality. Through disclosure and examination processes, a patent breadth  $y^*$  has been granted and become publicly known. If the firm subsequently develops a product of quality  $x + y$  with  $y \in [0, y^*]$ , then this product is perceived by PH as infringing upon the patent  $x$ . Quality  $x$  is just a basic research outcome and has no market value *per se*.<sup>8</sup> We analyze the cases where a commercially profitable product with

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<sup>7</sup> Note that in Gallini (1992) imitation resulting from inventing around is costly but always non-infringing.

<sup>8</sup> This type of invention fits into the categories of “research tools” (Schankerman and Scotchmer, 2001) and “essential inventions” (Encaoua et al., 2006) distinguished in the patent literature. As Chou and Haller (2007) find that at the industry level, it is a rather typical aspect of R&D in the sense that a

quality  $y$ , where  $x \leq y \leq y^*$ , has been developed by the firm and thus infringement occurs.<sup>9</sup> The cost of developing quality  $y$  is  $c_y$ . Once developed, the new product can be produced at zero cost and has potential market value  $\pi_y$ .

The crucial elements of patent litigation in our model are as follows: We assume for ease of notation that each party incurs the same litigation cost  $L > 0$ . An objective probability  $f(y)$  of PH winning the litigation is assumed to depict imperfect patent protection.<sup>10</sup> While both parties may agree privately whether or not an infringement occurs, the court may come to a different judgment. Occasionally, we treat  $y$  as variable and  $f(y)$  as a decreasing function of  $y \in [0, y^*]$  with  $f(0) = 1$  and  $f(y^*) = 0$ . The further away from  $x$  a new invention is, the less likely will it be adjudicated as infringing. If the court finds infringement in favor of the patent holder (the plaintiff), it assesses the extent of true damage  $(y)$ , and orders compensation in the form of a reasonable royalty,  $R \geq 0$ . According to the “analytical method” (see appendix A) and the model parameters,  $R$  would amount to the actual profit made by the infringer, that is,  $R = \pi_y - c_y$ .<sup>11</sup>

We model the strategic interaction as a **strategic game** between PH and the firm. The game lasts one period, which is defined as the time interval beginning when PH initiates the licensing offer and concluding when the infringement and damage award issues are resolved. The two players take several steps during the period. There is no discounting within the period.

Both players enter the game with exogenously given and commonly known  $y$ . PH, as a first mover, makes a licensing agreement offer simply by specifying  $S$  with  $S \in [0, \infty)$ , where  $S$  is a fixed-fee royalty:  $S$  represents the amount to be paid by the firm for the right to profit from the commercialization of the PH-sanctioned technology  $y$ . By proposing  $S = 0$ , PH tolerates the infringement without legal

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commercially profitable innovation often results from basic research.

<sup>9</sup> Although Lemley and Shapiro note that a very small portion of patents are ever litigated, they stress the importance of evaluating the incentive effects of a probabilistic patent litigation process. Our model and analysis focus on this particular infringement aspect.

<sup>10</sup> Various definitions of “imperfect protection” have been proposed in the patent literature. Waterson (1990) employs a “court cost function” comprising of litigation costs and damage fees awarded to the patent holder to depict uncertainty in patent infringement litigation. In a sequential framework, Llobet (2003) also adopts this particular view. In addition, he assumes that the patent holder has private information about the size of the innovation and, consequently, the probability of winning in litigation. Crampes and Langinier (2002) consider patents as imperfect protection against entry if the patent holder cannot observe infringement, cannot identify the infringer, or cannot afford costly enforcement. Anton and Yao (2007) also develop a model allowing for uncertainty whether the infringement is detected.

<sup>11</sup> This is also compatible with what Anton and Yao (2007) describe as the “best case” enforcement regime.

recourse. Knowing the proposed licensing offer, the firm has three strategic alternatives: (i) quit the project; (ii) pay the royalty proposed by PH; (iii) challenge the patent infringement allegation. In the latter contingency, PH has to make one more move: take no action or litigate. In accordance with U.S. practice and the doctrine of “lost profit,” we assume that the then-found-infringing firm retains the profit from this application while paying its litigation costs plus lost royalties.<sup>12</sup> Figure 1 summarizes the **extensive form** of the game, showing the order of decisions and the resulting (expected) payoffs.<sup>13</sup>

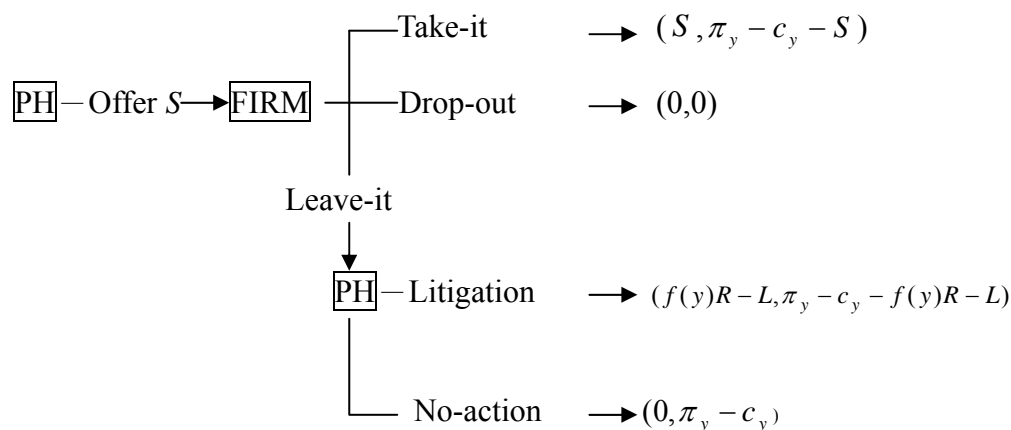


Figure 1: The Extensive Form

Let  $M = \{\text{No-action, Litigation}\}$  and  $N = \{\text{Take-it, Leave-it, Drop-out}\}$ . Then the **normal form** of the game has strategy spaces  $S_{PH} = \mathfrak{R} \times M^{\mathfrak{R}}$  for PH and  $S_F = N^{\mathfrak{R}}$  for the firm. We consider strategy pairs that are Nash equilibria, i.e. each player chooses a strategy that maximizes its expected payoff given the other player’s strategy. Moreover, we require **subgame perfection**: Equilibrium pairs of strategies induce equilibrium play in all subgames.

We distinguish four types of pure strategy equilibria. We note that which type emerges as the subgame perfect equilibrium of the game depends on the numerical

<sup>12</sup> The payoff structure and the order of bargaining-then-litigating events described here is similar to what Schankerman and Scotchmer (2001:206, Figure 1) consider in their “licensing a research tool” game. However, since their interest is not in assessing probabilistic patents, in the infringement subgame with a given liability damage rule, firm 2 in their model is always deemed infringed if it stays in the market, that is,  $f(y) = 1$ , for  $y \in [0, y^*]$  in our notation.

<sup>13</sup> One crucial difference to Chou and Haller (2007) lies in the award to a winning PH. Chou and Haller (2007) stipulate that PH receives the amount he asks for, i.e.  $S$  in our present notation. Notice that in equilibrium, PH never receives more than  $\pi_y - c_y$ , since the firm has the option to drop out.



specification of the model.<sup>14</sup>

1. The **Take-it equilibrium** is characterized by an offer  $S_t$  with

$$\pi_y - c_y - S_t \geq 0,$$

$$\pi_y - c_y - S_t \geq \pi_y - c_y - f(y)R - L, \text{ and}$$

$$f(y)R - L \geq 0.$$

The firm responds with Take-it to this offer. Should the firm play Leave-it in response to this offer, then PH would counter with Litigation.

2. The **Leave-it equilibrium** is characterized by an offer  $S_l$  with

$$\pi_y - c_y - f(y)R - L \geq 0,$$

$$\pi_y - c_y - f(y)R - L \geq \pi_y - c_y - S_l, \text{ and}$$

$$f(y)R - L \geq 0.$$

The firm responds with Leave-it to this offer and PH counters with Litigation.

3. The **No-Action equilibrium** is characterized by the damage award  $R$  with

$$f(y)R - L < 0, \text{ and}$$

$$\pi_y - c_y \geq 0.$$

The firm responds with Leave-it to all offers  $S > 0$  and PH counters with No-action.

4. The **Drop-out equilibrium** is characterized by an offer  $S_d$  with

$$\pi_y - c_y - S_d \leq 0,$$

$$f(y)R - L \geq 0, \text{ and}$$

$$\pi_y - c_y - f(y)R - L \leq 0.$$

The firm responds with Drop-out to this offer. If the firm responded with Leave-it to this offer, then PH would counter with Litigation.

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<sup>14</sup> PH, as a leader in this game, has the sole interest in proposing the offer  $R$  so as to collect the highest possible profit share from the firm. The No-Action and Drop-out equilibria where PH cannot generate any positive gain are therefore not the focus of the subsequent analysis. However, in a more complicated many-firm setting, these potential types of equilibria may impact upon PH's decision-making.

We proceed under the following simplifying assumption<sup>15</sup>:

**(A1)**  $\pi_y = a \cdot y; c_y = c \cdot y$  where  $a$  and  $c$  are constants satisfying  $a > c \geq 0$ .

Notice that constant marginal revenue occurs in a standard vertical (quality) differentiation problem. There consumers have utility functions of the form  $U = \theta y - p$  where  $\theta$  is a taste parameter and  $p$  is the price charged for the product of quality  $y$ . The distribution of tastes across consumers is given by the uniform distribution along the interval  $[0, \theta]$  with  $\theta > 0$ . Then, given  $y$ , the firm maximizes its gross profit by choosing the price level  $P_y = y\theta/2$ . The resulting gross profit is  $\pi_y = y\theta^2/4$ . Put  $a = \theta^2/4$ .

### III. EQUILIBRIUM ANALYSIS

We first characterize the types of equilibrium of interest. Specifically, the interval  $(0, y)$  can be partitioned into three areas each of which corresponding to a particular type of equilibrium. Necessary and sufficient conditions for these three types of equilibrium outcomes are provided in Proposition 1.

**Proposition 1.** (i) A Complete-Profit-Transfer Take-it equilibrium emerges, i.e., PH can extract all the profit it facilitates from the firm if and only if

- (a)  $y - yf(y) \leq L/(a - c)$  and
- (b)  $yf(y) \geq L/(a - c)$ .

(ii) A No-Action equilibrium emerges, i.e., the firm can retain all the profit if and only if

- (c)  $yf(y) < L/(a - c)$

(iii) An Incomplete-Profit-Transfer Take-it equilibrium emerges, i.e., a licensing offer  $S_i^i = (a - c)yf(y) < \pi_y - c_y$  is proposed and accepted if and only if

- (b)  $yf(y) \geq L/(a - c)$  and
- (d)  $y - yf(y) > L/(a - c)$ .

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<sup>15</sup> Since PH's sole interest is in maximizing the profit transferrable from the firm, he has no incentive to make an offer  $S > \pi_y - c_y$ . In other words, {Drop-out, Litigate} or {Drop-out, No-Action} can never be an equilibrium outcome since Drop-out is dominated by Take-it for the firm.

Several forces from various sources are at work in driving the above equilibrium configuration. Firstly, inequalities (a) and (d) determine how large a licensing fee would make the firm indifferent between acceptance and litigation. Secondly, inequalities (b) and (c) convey to what extent PH's threat to litigate is credible. Some simple comparative statics can help develop further the intuition for Proposition 1.

To begin with, suppose  $a, c, y$ , and  $L$  are parameterized such that  $y \geq L/(a-c)$  and  $f(y)$  is treated as a variable. Note that then (a) and (b) are simultaneously satisfied if and only if  $f(y)$  is sufficiently large (or alternatively speaking, the degree of second generation innovation is sufficiently low). To achieve a full-profit-transfer equilibrium with a licensing offer  $S_l = (a-c)y$ , PH's threat of litigation in the Leave-it subgame has to be credible, that is,  $f(y)R - L > 0$ . Now  $y \geq L/(a-c)$  can be translated as  $(a-c)y > L$ . Therefore, if  $R = (a-c)y$ , then the inequality  $f(y)R - L > 0$  holds trivially for the boundary case where  $f(y) = 1$ . By continuity, a high and only a high winning probability  $f(y)$  sustains the full rent extraction equilibrium. Moreover, we observe that (a) and (b) together imply  $f(y) \geq 1/2$ . This, however, reveals that a very innovative  $y$  associated with a low winning probability for PH will not result in full rent extraction.

Conversely, suppose  $a, c, y$ , and  $L$  are parameterized such that  $y < L/(a-c)$ . Then (a) is always satisfied while (b) breaks down for all  $f(y) \in [0,1]$ . That is, a low degree of innovation (small  $y$ ) associated with a high winning probability for PH will not sustain a full-profit-transfer equilibrium either. In a similar vein, an educated guess is that the existence of an incomplete-profit-transfer Take-it equilibrium may hinge upon some "intermediate"  $f(y)$ .

And it becomes immediately clear that when  $y \geq L/(a-c)$ , only a sufficiently small  $f(y)$  could sustain a No-Action equilibrium. Conversely but not surprisingly, for  $y < L/(a-c)$ , a No-Action equilibrium can be well expected since (d) holds trivially. To sum up, the preceding comments suggest that there might exist specific bounds for those intervals of  $y$  which permit a specific type of equilibrium.

To extend the analysis further and derive such bounds, we next consider a situation with exogenously given  $L, a, c, y^*$ , and a function  $f : [0, y^*] \rightarrow [0,1]$  satisfying:

**(A2)**  $f(0) = 1, f(y^*) = 0$ , and  $f$  is twice differentiable with  $f' < 0, f'' \leq 0$ .

(A2) describes PH's winning probability function which is decreasing in the innovation parameter  $y$ . A less obvious but equally plausible property exhibited in (A2) is that PH's winning probability drops faster as  $y$  moves further away from

$x = 0$ , or the closer  $y$  approaches the delimiter of patent protection,  $y^*$ . From a different perspective, *ceteris paribus*, the firm is enjoying “increasing returns to litigation” (Chou and Haller, 2007) as  $y$  improves.<sup>16</sup>

For expositional purposes, we denote  $\bar{y}$  the “median” of  $f(y)$ , i.e.  $\bar{y}$  is implicitly given by the condition  $f(\bar{y}) = 1/2$ .

**Proposition 2.** *Suppose*

$$(M) \quad \bar{y}/2 \geq L/(a-c).$$

*Then there exist  $y_l$ ,  $y_m$  and  $y_r$  with the following properties:*

- (i)  $0 < y_l \leq y_m \leq y_r < y^*$ ;
- (ii) *The highest-yield equilibrium for PH is a complete-profit-transfer Take-it equilibrium iff  $y \in [y_l, y_m]$ .*
- (iii) *A No-Action equilibrium emerges iff  $y \in [0, y_l) \cup (y_r, y^*]$ .*
- (iv) *The highest-yield equilibrium for PH is an incomplete-profit-transfer Take-it equilibrium with  $S_i^i = (a-c)yf(y) + L$  iff  $y \in (y_m, y_r]$ .*

**Corollary 1.** *Denote  $\hat{y} = \arg \text{Max}_{y \in [0, y^*]} yf(y)$ .*

- (i) *When  $L/(a-c) \in (\hat{y}f(\hat{y}), \infty)$  there only exist No-Action equilibria for  $y \in [0, y^*]$ .*
- (ii) *When  $L/(a-c) \in (\bar{y}/2, \hat{y}f(\hat{y})]$  there exist both Take-it and No-Action equilibria for  $y \in [0, y^*]$ .*

Note that even when litigation is not that costly relative to the profit of the commercialized product, the “full strength” patent protection exists only in the interval  $[y_l, y_m]$ . Both low and high end improvements would not induce actual patent litigation (No-Action). Because in both cases where “low  $y$  induced high  $f(y)$ ” or “high  $y$  induced low  $f(y)$ ”, PH’s expected gain from litigation is less than his litigation cost. In our model, as a consequence, the firm has an incentive (if it has a choice) to invent around with close imitation or invent enough with a quite

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<sup>16</sup> Waterson (1990), the seminal paper in the “signpost” literature, implicitly imposes an property on the “court cost function” which is equivalent to concavity of  $f(y)$ .

novel (though still infringing) product because of the lack of effective patent protection in these regions.<sup>17</sup>

It is obvious that our conclusions about the division of profit rely crucially on the relevant range for  $L/(a-c)$ . Under (M), with a relatively low  $L/(a-c)$ , three types of equilibria may emerge whereas a relatively high  $L/(a-c)$  would only allow No-Action equilibria for all  $y$  in the statutory claimed range of patent protection. Generally speaking, two counteracting factors drive PH's expected profit transfer in the analysis so far: what the firm is capable of, i.e., the magnitude of  $y$ , and what the odds are for PH to win the infringement case, i.e., the range of  $f(y)$ . Thus hypothesis (M) and the conditions in Corollary 1 delineate the extent to which PH can master this balancing act and extract profit from the firm.

## IV. Comparative Statics

### IV.1. Efficacy of patent protection in terms of complete profit transfer

In this sub-section we focus on the comparative statics with respect to several key variables - within their most interesting range in our set-up. In the sequel we denote  $\lambda \equiv y_m - y_l$  the length of the interval of  $y$  where a complete-profit-transfer Take-it equilibrium can be obtained.  $\lambda$  may serve as a measure of the efficacy of the patent system. We first investigate how  $\lambda$  is responding to variations of the litigation cost  $L$ , the gross profit parameter  $a$  and the product development cost parameter  $c$ . It suffices to see how  $L$  depends on the compound parameter  $k \equiv \frac{L}{a-c}$ .

**Lemma 2.** *Suppose  $\frac{\bar{y}}{2} \neq \hat{y}f(\hat{y})$ . Then  $\frac{\partial \lambda}{\partial k} < 0$  for  $k \in (\bar{y}/2, \hat{y}f(\hat{y}))$ . Moreover, the corresponding intervals  $[y_m(k), y_r(k)]$  are strictly nested.*

Intuitively, a higher litigation cost should have cast a stronger deterring effect upon the opportunistic firm turning down the offer. However, Lemma 2 says that even when the litigation cost is in the “favorable” range where a complete-profit-transfer

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<sup>17</sup> Empirically, as Mansfield et al (1981) point out, 60% of all patented and successful innovations were imitated within 4 years after introduction. Similar observations are also reported in Levin et al (1987).

Take-it equilibrium can be assured, higher litigation cost will diminish the effective patent protection measured by  $\lambda$ . Lemma 3 completes the examination of  $\lambda(k)$ .

**Lemma 3.**  $\lambda(k)$  is strictly concave in  $k$  and there exists a unique  $\tilde{k} \in (0, \bar{y}/2)$  such that  $\lambda(\tilde{k}) \geq \lambda(k)$  for all  $k \in R_+$ .

In a second type of comparative statics, we investigate how  $\lambda$ , the proposed patent protection measure, is affected by a change of patent protection. Intuition may suggest that the best way to help PH transfer profit from the firm is to grant PH a broad patent protection. Interesting enough, this is a premature conclusion as the next proposition shows. One more simplifying assumption, (A3) is imposed to establish the result. Prior to that we have to extend the model appropriately by postulating that the winning probability takes the more general form  $f(y; y^*)$ ,  $0 \leq y \leq y^*$ , where the patent breadth  $y^* > 0$  is treated as variable in the sequel. The extended notation  $\lambda(k; y^*)$ ,  $\tilde{k}(y^*)$ , etc. will be used.

(A3)  $f(y; y^*) = f(y/y^*; 1)$ , i.e.,  $f(y; y^*)$  is homogeneous of degree 0.

(A3) stipulates that the winning probability for PH depends only on the ratio  $y/y^*$ , not on the absolute magnitude of  $y$  or  $y^*$ . An extremely high  $y^*$  might correspond to a very broad claim such as “All non-human transgenic mammals” or “all hand-use calculators.” It is well documented (Lemley and Shapiro, 2005) that a typical defendant’s strategy of countering infringement litigation is to argue that the patent is “invalid” in the sense that PTO has failed to identify the existence of “prior art” in the original claim. Hence it appears reasonable that a broader set of claims leaves more room for purporting invalid patents.

Thus it is of economic significance to investigate how the specific features of probabilistic patents, i.e. the signpost interpretation of the patent system, respond to variations in patent breadth. In the present paper, imposing (A3) is a simple attempt to capture the effect of broadening patent protection, be it through a change of the written law or the doctrine of equivalents, and the reverse doctrine of equivalents.<sup>18</sup> Our focus here is to evaluate the impact upon the PH’s incentive to innovate from adjusting the patent protection under the doctrine of equivalents or other similar traits.

**Lemma 4.** Suppose  $k \in (0, \bar{y}/2)$ , then the functions  $y_l(k; y^*)$ ,  $y_r(k; y^*)$  and

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<sup>18</sup> Notice that here we are not concerned with the controversy over the judicial standard for infringement analysis under the doctrine of equivalents, e.g. tests like “element-by-element” and “invention as a whole” See, for example, Lau (1989) and Merges (1992).

$\lambda(k; y^*)$  are homogeneous of degree 1 in  $(k; y^*)$ . The functions  $\hat{y}(y^*)$  and  $\bar{y}(y^*)$  are homogeneous of degree 1 in  $y^*$ .

Let us first state a result that conforms to intuition: as patent protection becomes broader,  $\lambda(k; y^*)$  increases, i.e. the size of the interval where PH can extract all the surplus increases.

**Proposition 4.** *The following three assertions hold:*

- (i)  $\frac{\partial}{\partial k} \lambda(k; y^*)$  is strictly increasing in  $y^* > 0$  as long as  $0 < k < \bar{y}(y^*)/2$ .
- (ii)  $\lambda(k; y^*)$  is strictly increasing in  $y^* > 0$  as long as  $0 < k < \bar{y}(y^*)/2$ .
- (iii)  $\tilde{k}(y^*)$  is strictly increasing in  $y^* > 0$ .

Let us now proceed to the promised, somewhat less intuitive result: As patent protection becomes broader, the relative size of the interval where PH can extract all the surplus decreases.

**Proposition 5.** *For any  $0 < y^* < y^{**}$ , there exists  $k(y^*, y^{**}) > 0$  such that  $\frac{\lambda(k; y^{**})}{y^{**}} < \frac{\lambda(k; y^*)}{y^*}$  for all  $0 < k < k(y^*, y^{**})$ .*

Proposition 5 states that even though  $\lambda(k; y^*)$  increases as patent protection becomes broader,  $\lambda / y^*$ , that is the relative size of the interval where PH can extract all the surplus, may be falling for certain  $k$ . In other words, the efficacy measure of patent enforcement defined as the fraction of the infringing  $y$  that provide maximal incentive for PH to innovate, can apparently diminish when the court employs the doctrine of equivalents and imperfect patent protection exhibits such feature as (A3).

## IV.2. Efficacy of patent protection in terms of average expected profit transferred

In this sub-section we focus on the comparative statics with respect to  $(k; y^*)$ . In the sequel we denote

$$E\pi_p \equiv \frac{1}{y^*} \left\{ \int_{y_l}^{y_m} (a-c)y dy + \int_{y_m}^{y_r} [(a-c)yf(y) + L] dy \right\}.$$

$E\pi_p$  is the expected payoff for PH given that  $y$  is uniformly distributed along the interval of proclaimed patent protection  $[0, y^*]$ . Recall that in Proposition 1, we have shown that the interval  $[0, y^*]$  can be partitioned into three areas each of which is characterized by a particular type of equilibrium. The first integral derives from the area that gives rise to the complete-profit-transfer Take-it equilibrium. The second integral derives from the area that gives rise to the incomplete-profit-transfer Take-it equilibrium with  $S_t^i = (a-c)yf(y) + L$ . In the third area, PH obtains zero profit from the No-Action equilibrium.

We first investigate how  $E\pi_p$  is responding to variations of  $k$ .

**Lemma 5.** *Suppose  $\bar{y}/2 \neq \hat{y}f(\hat{y})$ , then  $\frac{\partial E\pi_p}{\partial k} < 0$  for  $k \in (\bar{y}/2, \hat{y}f(\hat{y}))$ .*

*Moreover the corresponding intervals  $[y_m(k), y_r(k)]$  are strictly nested.*

**Lemma 6.** *Suppose  $\bar{y}/2 = \hat{y}f(\hat{y})$ , then there exists a unique  $\overset{\circ}{k} \in (\tilde{k}, \bar{y}/2)$  such that  $E\pi_p(k)$  is decreasing in  $k \in [k, \bar{y}/2]$ .*

Lemma 5 and 6 are reminiscent of Lemma 2 and 3. In particular, the properties of  $\lambda(k)$  and  $E\pi_p$  follow directly from the concavity of  $f(y)$ , the probability of PH winning in litigation. In other words, the phenomenon observed here, that is, shrinkage of the interval of complete profit transfer and decrease in PH's average expected profit with respect to certain  $k$ , can be attributed to a particular aspect of imperfect patent protection, namely increasing returns to litigation for the firm.

**Lemma 7.** *The efficiency ratio  $\frac{E\pi_p(k; y^*)}{E\pi(k; y^*)}$  is homogeneous of degree zero in  $(k; y^*)$ .*

Lemma 7, comparable to Proposition 5, reminds us of the potential drawback of an expanded patent protection: an increase in the relative profit transferred to PH as a result of broadened patent breadth may not improve the efficacy of patent protection by some measure.<sup>19</sup>

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<sup>19</sup> Note that the application of the doctrine of equivalents may be associated with litigation cost in the



Some caution is warranted in interpreting Lemma 7. Namely, the result admits two alternative interpretations with respect to the average expected profit transferred from the firm to PH when patent breadth is adjusted. First, observe that  $E\pi(k; y^*)$ , the average profit to PH under perfect patent protection with  $y$  uniformly distributed over  $[0, y^*]$  is just  $(a-c)\frac{y^*}{2}$ . Now suppose that under two patent protection regimes  $y^*$  and  $\lambda y^*$  (w.l.o.g,  $\lambda > 1$ ), the innovation parameter  $y$  is distributed with uniform density  $\frac{1}{y^*}$  on  $[0, y^*]$  and with density  $\frac{1}{\lambda y^*}$  on  $[0, \lambda y^*]$ . Set

$\frac{E\pi_p(k; y^*)}{E\pi(k; y^*)} = \alpha < 1$ . Then Lemma 7 implies that

$$E\pi(\lambda k; \lambda y^*) = \alpha E\pi_p(\lambda k; \lambda y^*) = \alpha(a-c)\frac{\lambda y^*}{2} > \alpha(a-c)\frac{y^*}{2} = E\pi_p(k; y^*)$$

Alternatively, if the patent protection regime  $y^*$  can be viewed as a continuous contraction from the  $\lambda y^*$  regime, then the innovation parameter  $y$  should be distributed with a uniform density  $\frac{1}{\lambda y^*}$  on  $[0, y^*]$  and  $[0, \lambda y^*]$ . Then Lemma 7 implies

$$E\pi(\lambda k; \lambda y^*) = \alpha E\pi_p(\lambda k; \lambda y^*) = \alpha(a-c)\frac{\lambda y^*}{2} > \alpha(a-c)\frac{y^*}{2} \cdot \frac{y^*}{\lambda y^*} = E\pi_p(k; y^*).$$

While the former interpretation is better suited for the description of an adjustment in written law, the latter is more compatible with the application of the reverse doctrine of equivalents. In either case, PH is enjoying an increase in his absolute average profit transferred from the infringing firm under a broader patent protection.

## V. Concluding Remarks

In this article, we investigate the division of profit between a patent holder and a derived product producer in an environment with uncertainty about the outcome of infringement litigation but with certainty about the damage: a reasonable royalty granted to the prevailing plaintiff. Our analysis identifies the conditions on model parameters that permit a complete-profit-transfer Take-it equilibrium, an incomplete-profit-transfer Take-it equilibrium or a No-Action equilibrium, respectively. The interval of the proclaimed patent protection  $[0, y^*]$  can be partitioned into three areas each of which is characterized by a particular type of

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following sense: (1) enforcement cost goes up as  $y^*$  is increased, and (2) the litigants have more burden of proof.

equilibrium. Complete profit transfer gives a patent holder maximal incentive to innovate. A subsequent innovator may want to imitate either with a close substitute or with a much advanced but still infringing product, to reach a No-Action equilibrium with zero profit transfer. Comparative statics with respect to important policy parameters such as litigation costs and patent breadth is also performed.

In a broader context, while some of the litigation literature assumes litigation to always be profitable for the plaintiff, to rule out “nuisance” suits (Bebchuk (1984); Reinganum and Wilde, 1986), the current model accommodates inaction as one of the strategies the patent holder may choose (Nalebuff, 1987; Meurer, 1989). That is, a patent holder may not always be willing to use the court system. Consequently, credibility of threats has a significant effect on the equilibrium outcome. Recall that which type of equilibrium will prevail depends crucially on (a) how large a licensing fee would make firm indifferent between acceptance and litigation and (b) how credible PH’s threat is to litigate in case the infringing firm were to turn down the offer. To maintain a credible litigation threat the patent holder must evaluate the potential gain and loss when the infringing firm refuses to settle. The issue of credibility may severely restrict the patent holder’s capacity of achieving the desired outcome - complete profit transfer. In addition, the particular phenomenon observed from the above comparative statics, that is, shrinkage of the interval of complete profit transfer and decrease in PH’s average expected profit with respect to certain  $k$ , is again a result of the dominance of effect (b) over effect (a).

Our analysis pays special attention to the effects of the variations in patent breadth, either by an adjustment in the written law or by application of the doctrine of the equivalents and the reverse doctrine of equivalents. It is clear that the current paper is not suggesting the “optimal” patent system. Rather it aims at assessing the efficacy of the current patent system when uncertainty about the outcome of infringement litigation is taken into consideration. In particular, we are concerned with the incentive to innovate, not the harm (deadweight loss) caused by blocking patents as Merges and Nelson (1992) have emphasized. Therefore, it is not our intention to determine which  $y$  should be allowed to escape the “web of infringement” (Scotchmer, 1996).

## Appendix A

**A Brief Review of the Doctrines on Reasonable Royalty.** Prior to 1946, a successful patent claimant could choose between the amount of damages she suffered and the amount of profits earned by the infringer (Chisum, 1980). However the high cost of determining an infringer's profits eventually led Congress to drop infringer profits as an alternative measure of recovery [Act of August 1, 1946, ch. 726 s 1, 60 stat. 778 (current version at §35 U.S.C. s 284 (1952))]. More sophisticated doctrines have been applied since. Despite the revision, an infringer's profits continue to be crucial elements in computing the patent holder's damages - either as an approximation for the patent holder's "lost profits" or as a factor in determining a reasonable royalty. In general, a successful patent claimant is entitled to recover the profits she would have made but for the infringement; if lost profits cannot be proven, she is entitled to a reasonable royalty.

### (A) *Lost Profits:*

(1) The Cornerstone case for lost profits: *Panduit Corp. v. Stahl Bros. Fiber Works, Inc.* 575 F.2d 1152, 197 U.S.P.Q. (BNA) 726 (6th Cir.1978).

Four factors enumerated in *Panduit* have been used as the primary guidelines for determining whether a patent holder is entitled to recover lost profits.

### (B) *Reasonable Royalty:*

(1) The Benchmark case: *Georgia-Pacific Corp v. United States Plywood Corp.* 318 F. Supp. 1116, 166 U.S.P.Q. (BNA) 235 (S.D.N.Y. 1970), modified, 446 F.2d 295, 170 U.S.P.Q. 369 (2d Cir.), cert. denied, 404 U.S. 870 (1971).

*Georgia-Pacific* has been relied upon heavily for its fifteen factors, among others, to be verified in determining a reasonable royalty.

(2) Variation of *Georgia-Pacific*: *Hanson v. Alpine Valley Ski Area, Inc.* 718 F.2d 1075, 219 U.S.P.Q. (BNA) 679 (Fed. Cir.1983).

In *Hanson* the Federal Circuit has stated: 'The reasonable royalty may be based upon,..., a hypothetical royalty resulting from arm's length negotiation between a willing licensor and a willing licensee.'

Conceivable problems with this hypothetical negotiation are manifold: First, the court is required to reconstruct a "fancy contract" based upon fantasy and flexibility [*Fromson v. Western Litho Plate & Supply Co.* 853 F.2d 1568, 1575-76, 7 U.S.P.Q.2d

1606 (Fed Cir. 1988)]. Secondly, it ignores the adverse impact upon converting property rule into liability rule at random wills [See Calabresi and Melamed (1972)]. That is, a flat reasonable royalty with no punitive effect would have reduced the incentive to innovate in the first place. Though an infringer may not be completely indifferent between an ex-ante licensing agreement and the ex-post damage award since there may be conceivable loss of goodwill and substantial sunk costs.

(3) What flows directly from the willing licensor/willing licensee model is the 'analytical method' (or accounting method). It computes a reasonable royalty based on the infringer's pre-infringement projection of profits. See, for instance, *TWM Mfg. Co. v. Dura Corp.*, 789 F.2d 895, 229 U.S.P.Q. (BNA) 525 (Fed. Cir. 1986).

In reference to the current modeling approach, the following features are worth further investigation.

To begin with, to be eligible for the lost profits compensation, the patent holder has to show she has capacity to produce the volume of potential sales lost due to infringement. However, no stringency has been demonstrated in this respect by the courts. For example, to prove its capacity to make the infringer's sales, the patent holder does not have to show that it had a plant ready and existing [See *Livesay Window Co. v. Livesay Industries, Inc.*, 251 F.2d 469, 473, 116 U.S.P.Q. 167, 171 (5th Cir. 1958); *W. L. Gore & Associated, Inc. v. Carlisle Corp.*, 198 U.S.P.Q. 353, 367 (D. Del. 1978)]. The court even considered the possibility of subcontracting as the patent holder's potential capacity (see *Gyromat Corp. v. Champion Spark Plug Co.*, 735 F.2d 549, 554 222 U.S.P.Q. 4, 7 (Fed. Cir. 1984)). As a result, the patent holder in our model, though without marketing power, would still be entitled to lost profits, the amount of which may be determined according to the infringer's actual profits.

Secondly, despite the fact that the doctrines and their variations on reasonable royalty have been widely stated in patent damages cases, the analytical method seems to be the dominant guiding principle for computing reasonable royalty [See Conley (1987) citing that the courts giving only 'lip service to the willing licensor/willing licensee model']. Furthermore, as a simple rule, the courts just subtract the infringer's usual profit from the profit earned by the infringement, and award the entire difference to the patent holder. In a sense the analytical approach is a return to awarding to the patent holder the infringer's profits from the use of the invention.

Note that in the context of our model, the value of the second generation product is completely attributable to the basic patented technology. Thus the doctrines of 'entire market value' and 'apportionment' would yield the same compensation figure for the

patent holder [See, e.g. Westinghouse v. Wagner, §225 U.S. 604, 614 (1912)]. Specifically, in terms of the model parameters, the analytical approach amounts to a reasonable royalty  $R = (a - c)y$  as we assume.

## APPENDIX B

### Proof of Proposition 1:

(i) For a licensing agreement to prevail, i.e., a Take-it equilibrium to exist, the following conditions are necessary and sufficient:

$$\pi_y - c_y - S_t \geq 0, \pi_y - c_y - S_t \geq \pi_y - c_y - f(y)R - L, \text{ and } f(y)R - L \geq 0.$$

We first explore the possibility that PH can extract all the profit from the firm, i.e., where an offer  $S = (a - c)y$  gets accepted in equilibrium. Then the previous conditions are equivalent to:

$$S = (a - c)y, \tag{1}$$

$$R \leq \frac{L}{1 - f(y)} \tag{2}$$

$$R \geq \frac{L}{f(y)} \tag{3}$$

(2) and (3) are then equivalent to

$$y[1 - f(y)] \leq \frac{L}{a - c} \text{ and} \tag{4}$$

$$yf(y) \geq \frac{L}{a - c}. \tag{5}$$

(ii) We proceed with the necessary and sufficient conditions for a No-Action equilibrium, i.e.,  $\pi_y - c_y \geq 0$ , and  $f(y)R - L \leq 0$ . With (A1) they are equivalent to:

$$\pi_y - c_y \geq 0, \text{ and } f(y)R - L \leq 0 \text{ which in turn is equivalent to}$$

$$yf(y) \leq \frac{L}{a - c}. \tag{6}$$

(iii) We commence with the necessity proof. Recall that for a licensing agreement to prevail, i.e., a Take-it equilibrium to exist, the following conditions must be satisfied:

$$\pi_y - c_y - S_t \geq 0,$$

$$\pi_y - c_y - S_t \geq \pi_y - c_y - f(y)R - L, \text{ and}$$

$$f(y)R - L \geq 0.$$

Note that in an incomplete-profit-transfer Take-it equilibrium the firm enjoys positive payoff after the transfer of profit through a licensing agreement, i.e.,  $\pi_y - c_y - S_t \geq 0$ .

We then explore the possibility that PH can extract profit by offering  $S_t = (a - c)yf(y) + L$ , which gets accepted in equilibrium. With (A.1) and the assumption on reasonable royalty, the previous conditions can be rewritten as:

$$\pi_y - c_y - [(a - c)yf(y) + L] > 0, 0 \geq 0, \text{ and } yf(y) \geq \frac{L}{a - c}.$$

These are just conditions (d) and (b):

$$y[1 - f(y)] > \frac{L}{a - c}, \text{ and} \tag{d}$$

$$yf(y) \geq \frac{L}{a - c}. \tag{b}$$

Now we turn to the sufficiency proof. Suppose (b) and (d). Since only the Take-it or the Leave-it equilibrium has the potential for generating positive payoffs for PH, we will focus on these two types of equilibria. We start with the possibility for PH to extract profit via a Take-it equilibrium.

Define

$$S_t \equiv (a - c)yf(y) + L$$

Thus we can have

$$\pi_y - c_y - S_t \geq \pi_y - c_y - [(a - c)yf(y) + L] \tag{7}$$

Also, by (d), we can infer

$$\pi_y - c_y > (a - c)yf(y) + L \tag{8}$$

or

$$\pi_y - c_y - [(a - c)yf(y) + L] > 0, \text{ that is}$$

$$\pi_y - c_y - S > 0. \tag{9}$$

Next note that because of (A.1) and the assumption on the reasonable royalty, (b) can be rewritten as

$$f(y)R - L \geq 0. \tag{10}$$

(7), (9) and (10) establish that, indeed,  $S_t$  is a Take-it equilibrium offer. It is also obvious that such a licensing offer yields higher payoff for PH than any Leave-it equilibrium since  $S_t = (a - c)yf(y) + L > (a - c)yf(y) - L$ . We have shown that the combination of (b) and (d) is equivalent to the necessary and sufficient conditions for an Incomplete-Profit-Transfer equilibrium characterized by  $S_t = (a - c)yf(y) + L$ .

Notice that strict inequality in (d) implies strict inequality in (8), that is incomplete-profit-transfer *strictu sensu*. This completes the proof. ■■

**Proof of Proposition 2:** First, we perform comparative statics with respect to  $y \in [0, y^*]$ . For this purpose, we introduce the functions  $g_1(y) = y[1 - f(y)]$  and  $g_2(y) = yf(y)$  which appear in (4) and (5) and, obviously, play a critical role in our analysis. Notice that  $g_1' = 1 - f - y \cdot f' > 0$  and  $g_1'' = -2f' - y \cdot f'' > 0$  in the interval  $(0, y^*)$ . Hence  $g_1$  is strictly increasing and strictly convex in  $y$  with  $g_1(0) = 0$  and  $g_1(y^*) = y^*$ .

Further notice that  $g_2' = (y \cdot f)' = f + y \cdot f'$  and  $g_2'' = (y \cdot f)'' = 2f' + y \cdot f'' < 0$ . Thus  $g_2$  is strictly concave in  $y$  with  $g_2(0) = g_2(y^*) = 0$ . Consequently,  $g_2$  has a unique maximizer  $\hat{y}$  in  $(0, y^*)$ . This maximizer is given as the unique solution of the first order condition

$$g_2'(\hat{y}) = f(\hat{y}) + \hat{y}f'(\hat{y}) = 0.$$

**Claim:**  $\hat{y} \leq \bar{y}$  and  $0 < y^*/2 < \bar{y} < y^*$ .

To show this claim, recall that  $g_2(y)$  is strictly concave in  $y$  with  $g_2(0) = 0$  and  $g_2(y^*) = 0$ . By Takayama (1985) Theorem 1.C.3:  $f$  is concave on  $(0, y^*)$  if and only for any  $x, y \in (0, y^*)$ :  $f'(y) \cdot (x - y) \geq f(x) - f(y)$ .

Evaluate this inequality at  $y = \bar{y}$  and let  $x \rightarrow 0$ . Then  $-f'(\bar{y})\bar{y} \geq 1 - \frac{1}{2}$  or

$f'(\bar{y}) \cdot \bar{y} \leq -\frac{1}{2}$ . Adding  $f(\bar{y}) = \frac{1}{2}$  to the latter inequality yields

$g_2'(\bar{y}) = \bar{y} \cdot f'(\bar{y}) + f(\bar{y}) \leq 0$ . Strict concavity of  $g_2$  and  $g_2'(\hat{y}) = 0$  imply the assertion  $\hat{y} \leq \bar{y}$ . Further, (A2) has the immediate implication  $0 < y^*/2 < \bar{y} < y^*$ .

This concludes the proof of the claim.

Next, recall that  $g_1(y)$  is strictly increasing and strictly convex in  $y$  with  $g_1(0) = 0$  and  $g_1(y^*) = y^*$ . Hypothesis (M) amounts to  $g_1(\bar{y}) = g_2(\bar{y}) = \frac{\bar{y}}{2} \geq \frac{L}{a-c}$ .

**Part (i):**

Recall that  $\bar{y}$  is defined as the intersection point of  $g_1(y)$  and  $g_2(y)$ , i.e.,

$g_1(\bar{y}) = g_2(\bar{y}) = \frac{\bar{y}}{2}$ . By the hypothesis, the continuity and other properties of  $g_2$ , and

the intermediate value theorem, there exist  $z_l \in (0, \hat{y}]$  and  $z_r \in [\hat{y}, y^*)$  such that

$$g_2(\bar{y}) \geq g_2(z_l) = g_2(z_r) = \frac{L}{a-c}.$$

Next note that (M) implies  $y^* > g_2(\hat{y}) \geq \frac{L}{a-c} > 0$ . Then, by the continuity and other properties of  $g_1$  and the intermediate value theorem, there exists a unique  $z \in (0, y^*)$  with  $g_1(z) = \frac{L}{a-c}$ . We then compare the magnitudes of  $z_l$  and  $z$ .

Now recall that  $f(\bar{y}) = \frac{1}{2}$  and therefore  $\frac{L}{a-c} \leq \frac{\bar{y}}{2} = g_1(\bar{y}) = g_2(\bar{y})$ . Therefore,  $\bar{y} \in [z_l, z_r]$ , by the strict concavity of  $g_2$ . Also,  $z \leq \bar{y}$ , by the strict monotonicity of  $g_1$ . Hence  $z \leq z_r$ . Moreover,  $0 = g_1(0) = g_2(0)$ ,  $\frac{1}{2} \cdot \bar{y} = g_1(\bar{y}) = g_2(\bar{y})$ , strict convexity of  $g_1$  and strict concavity of  $g_2$  imply  $g_1(y) < \frac{1}{2} \cdot y < g_2(y)$  for

$y \in (0, \bar{y})$ . If  $\bar{y} = z_l$ , then  $z = \bar{y} = z_l$ . If  $\bar{y} > z_l$ , then  $g_1(z_l) < g_2(z_l) = \frac{L}{a-c}$ . Thus  $z > z_l$ . In any case, therefore,  $z \in [z_l, z_r]$ .

Now set  $y_l = z_l$ ,  $y_m = z$  and  $y_r = z_r$ . Then (i) is satisfied. •••

**Part (ii):**

We commence with the sufficiency proof. When condition (M) holds and  $y \in [y_l, y_m] = [z_l, z]$ , then  $y \in [z_l, z_r]$  and the strict concavity of  $g_2$  implies (b)  $yf(y) \geq \frac{L}{a-c}$ . Further  $y \in [y_l, y_m]$  implies  $y \leq z$ . Since  $g_1(y)$  is an increasing

function in  $y \in [0, y^*]$ , condition (a)  $y[1 - f(y)] \leq \frac{L}{a-c}$  holds as well.

Now we turn to the necessity proof. (b) implies that  $y \in [z_l, z_r]$ . (a) implies that  $y \leq z$ . Together (a) and (b) imply  $y \in [z_l, \min\{z_r, z\}] = [y_l, y_m]$ . Note that we know from Proposition 1 that by offering  $R_t = (a-c)y$ , PH can extract all the profit it facilitates from the firm if and only if (a) and (b) both hold. We have shown that under the hypothesis (c), the combination of (a) and (b) is equivalent to  $y \in [y_l, y_m]$ . •••

**Part (iii):**

We commence with the sufficiency proof. When condition (M) holds and  $y \in [0, y_l] \cup [y_r, y^*]$ , then  $y \in [0, y^*] \setminus [z_l, z_r]$  and the strict concavity of  $g_2$

implies (c)  $yf(y) \leq \frac{L}{a-c}$ .

Now we turn to the necessity proof. By the strict concavity of  $g_2$ , (c) implies



that  $y \in [0, y^*] \setminus [z_l, z_r]$  which in turn implies  $y \in [0, y_l] \cup [y_r, y^*]$ . Note that we know from Proposition 1 that a No-Action equilibrium is attainable if and only if (c) holds. We have shown that under the hypothesis (M), (c) is equivalent to  $y \in [0, y_l] \cup [y_r, y^*]$ . •••

**Part (iv):**

We commence with the sufficiency proof. When condition (M) holds and  $y \in (y_m, y_r]$ , then  $y \in (z, z_r]$  and the strict concavity of  $g_2$  implies (b)  $yf(y) \geq \frac{L}{a-c}$ . Further  $y \in [y_m, y_r]$  implies  $y > z$ . Since  $g_1(y)$  is an increasing

function in  $y \in [0, y^*]$ , condition (d)  $y[1 - f(y)] > \frac{L}{a-c}$  holds as well.

Now we turn to the necessity proof. (b) implies that  $y \in [z_l, z_r]$ . (d) implies that  $y > z$ . Together (b) and (d) imply  $y \in (y_m, y_r]$ . Note that we know from Proposition 1 that an incomplete-profit-transfer Take-it equilibrium is attainable if and only if (b) and (d) both hold. We have shown that under the hypothesis (M), the combination of (b) and (d) is equivalent to  $y \in [y_m, y_r]$ . This completes the proof. ■■

Proof of Lemma 2: Suppose  $a, c, y$ , and  $f(y)$  are given such that

$$k \in \left( \frac{\bar{y}}{2}, \hat{y}f(\hat{y}) \right). \text{ Together with the supposition } \frac{\bar{y}}{2} \neq \hat{y}f(\hat{y}) \text{ this implies that}$$

$g_2(\hat{y}) > k$  thus  $z_r > \hat{y} > z_l$ . It can also be inferred that  $\bar{y} \notin [z_l, z_r]$  by the strict concavity of  $g_2$  and  $z > \bar{y}$  by the strict monotonicity of  $g_1$ . By Lemma 1,  $\bar{y} \geq \hat{y}$  and  $\bar{y} \notin [z_l, z_r]$  imply  $\bar{y} > z_r$ .

Thus  $z > \bar{y} > z_r$ . So  $\lambda = y_m - y_l = \min[z_r, z] - z_l = z_r - z_l$ . The strict concavity and

the other properties of  $g_2$  imply that for all  $k_1, k_2$  such that  $\frac{\bar{y}}{2} < k_1 < k_2 < \hat{y}f(\hat{y})$

the corresponding  $z_r(k_1), z_l(k_1), z_r(k_2)$  and  $z_l(k_2)$  have the following order:

$$z_l(k_1) < z_l(k_2) < \hat{y} < z_r(k_2) < z_r(k_1) \text{ or} \\ \lambda(k_1) = [z_r(k_1) - z_l(k_1)] > [z_r(k_2) - z_l(k_2)] = \lambda(k_2).$$

This implies the assertion. ■■

We need a technical auxiliary result to proceed further:

**Lemma A1.** Suppose that  $k = g(y)$  is strictly increasing, concave (convex) and twice continuously differentiable in the non-empty interval  $(a, b)$  and suppose that  $g'(y) \neq 0$  for  $y \in (a, b)$ . Then  $y = g^{-1}(k)$  exists and is monotone, convex

(concave), and twice continuously differentiable with respect to  $k$ .

Proof: The existence, monotonicity, and twice continuous differentiability of  $g^{-1}$  are assured by the inverse function theorem; see Fleet (1966; Th.10.9.5). Moreover,

$$\text{we have } g^{-1'}(k) = \frac{1}{g'(g^{-1}(k))}.$$

Now, the only task left is to prove the concavity (convexity) conversion. Differentiation and application of the chain rule to the foregoing formula for  $(g^{-1})'$  yield

$$g^{-1''}(k) = -\frac{g''(g^{-1}(k))}{[g'(g^{-1}(k))]^3},$$

which has sign opposite to that of  $g''(g^{-1}(k))$ . This implies convexity (concavity) of  $g^{-1}(k)$ . ■■

Proof of Lemma 3: We consider three cases where  $\bar{k} = g_2(\bar{y}) = \frac{\bar{y}}{2}$ .

Case (i):  $k \in (\hat{y}f(\hat{y}), \infty)$ . Then trivially  $\lambda(k) = 0$  by Corollary 1-(i), that is, there does not exist a non-degenerate interval  $[y_l, y_m]$ .

Case (ii):  $k \in \left(\frac{\bar{y}}{2}, \hat{y}f(\hat{y})\right]$ . Then, by Lemma 2,  $\lambda(\bar{k}) \geq \lambda(k)$ . (This is, however, a bit more than what Lemma 2 states. When  $k = \hat{y}f(\hat{y})$ ,  $\lambda(k)$  is equal to zero since  $y_l$  and  $y_r$  coincide. So we include this boundary point in the statement.)

Case (iii):  $k \in \left(0, \frac{\bar{y}}{2}\right)$ . Since both  $g_1$  and  $g_2$  are continuous, monotone, and twice differentiable, by the inverse function theorem, the following functions are well defined, unique, and twice differentiable:

$$h_1(k) : \left[0, \frac{\bar{y}}{2}\right] \mapsto [0, \bar{y}] \text{ which } h_1(g_1(y)) = y \text{ for all } y \in [0, \bar{y}],$$

$$h_2(k) : \left[0, \frac{\bar{y}}{2}\right] \mapsto [0, \bar{y}] \text{ which } h_2(g_2(y)) = y \text{ for all } y \in [0, \bar{y}].$$

Furthermore, by Lemma A1,  $h_1$  is monotone and strictly concave while  $h_2$  is monotone and strictly convex. Therefore  $\lambda(k) = h_1(k) - h_2(k)$  is strictly concave in  $k$ . Notice that  $h_1'$  is continuously decreasing from  $h_1'(0) = \infty$  to  $h_1'(\bar{k}) = \frac{1}{g_1'(\bar{y})}$

and  $h_2'$  is continuously increasing from  $h_2'(0) = 1$  to  $h_2'(\bar{k}) = \frac{1}{g_2'(\bar{y})}$ . By Lemma 1

we already know that  $\bar{y} \geq \hat{y}$  which implies  $g'_2(\bar{y}) = f(\bar{y}) + \bar{y}f'(\bar{y}) \leq 0$ . Since

$$f(\bar{y}) = \frac{1}{2}, \quad \bar{y}f'(\bar{y}) \leq -\frac{1}{2} \quad \text{or} \quad -\bar{y}f'(\bar{y}) \geq \frac{1}{2}. \quad \text{Then}$$

$$g'_1(\bar{y}) = 1 - f(\bar{y}) - \bar{y}f'(\bar{y}) \geq 1 - \frac{1}{2} + \frac{1}{2} = 1. \quad \text{Thus} \quad \frac{1}{g'_1(\bar{y})} = h'_1(\bar{k}) \leq 1 < \frac{1}{g'_2(\bar{y})} = h'_2(\bar{k}).$$

Set  $H(k) = h'_1(k) - h'_2(k)$ .  $H$  is strictly decreasing and continuous with  $H(0) > 0$  and  $H(\bar{k}) < 0$ . By the intermediate value theorem, there exists a unique  $\tilde{k} \in \left(0, \frac{\bar{y}}{2}\right)$

such that  $H(\tilde{k}) = h'_1(\tilde{k}) - h'_2(\tilde{k}) = 0$ , that is  $\lambda'(\tilde{k}) = 0$ . By strict concavity of  $\lambda(k)$ ,

such a  $\tilde{k}$  will be the unique global maximizer of  $\lambda$  in  $k \in \left[0, \frac{\bar{y}}{2}\right]$ .

Cases (i), (ii), and (iii) together imply  $\lambda(\tilde{k}) \geq \lambda(k)$  for all  $k \in \mathfrak{R}_+$ . This completes the proof. ■■

**Proof of Lemma 4:** Consider  $\alpha > 0, y^* > 0$  and  $k \geq 0$ . Then:

$$\eta \in [\alpha y_l(y^*; k), \alpha y_r(y^*; k)]$$

$$\Leftrightarrow \eta = \alpha y \quad \text{and} \quad y \in [y_l(y^*; k), y_r(y^*; k)]$$

$$\Leftrightarrow \eta = \alpha y \quad \text{and} \quad y \cdot f(y; y^*) \geq k \quad \text{and} \quad y \cdot (1 - f(y; y^*)) \leq k$$

$$\Leftrightarrow \eta = \alpha y \quad \text{and} \quad y \cdot f(\alpha y; \alpha y^*) \geq k \quad \text{and} \quad y \cdot (1 - f(\alpha y; \alpha y^*)) \leq k$$

$$\Leftrightarrow \eta = \alpha y \quad \text{and} \quad \alpha y \cdot f(\alpha y; \alpha y^*) \geq \alpha k \quad \text{and} \quad \alpha y \cdot (1 - f(\alpha y; \alpha y^*)) \leq \alpha k$$

$$\Leftrightarrow \eta \cdot f(\eta; \alpha y^*) \geq \alpha k \quad \text{and} \quad \eta \cdot (1 - f(\eta; \alpha y^*)) \leq \alpha k$$

$$\Leftrightarrow \eta \in [y_l(\alpha y^*; \alpha k), y_r(\alpha y^*; \alpha k)].$$

This shows that in the relevant range,  $y_l(y^*; k)$  and  $y_m(y^*; k)$  and, consequently,  $\lambda(y^*, k)$  are homogeneous of degree 1 in  $(y^*, k)$ .

Moreover,  $f(\lambda \bar{y}(y^*); \lambda y^*) = f(\bar{y}(y^*); y^*) = \frac{1}{2}$  implies  $\bar{y}(\lambda y^*) = \lambda \bar{y}(y^*)$ .

Finally, (A3) implies  $f(y; \lambda y^*) = f(y/\lambda; y^*)$  and, hence,

$$\frac{\partial}{\partial y} f(y; \lambda y^*) = \frac{1}{\lambda} \frac{\partial}{\partial y} f(y/\lambda; y^*). \quad \text{Therefore,} \quad f(\hat{y}(y^*); y^*) + \hat{y}(y^*) \cdot \frac{\partial}{\partial y} f(\hat{y}(y^*); y^*) = 0 \quad \text{if}$$

$$\text{and only if} \quad f(\lambda \hat{y}(y^*); \lambda y^*) + \lambda \hat{y}(y^*) \cdot \frac{\partial}{\partial y} f(\lambda \hat{y}(y^*); \lambda y^*) = 0.$$

That means  $\hat{y}(\lambda y^*) = \lambda \hat{y}(y^*)$ . ■■

Proof of Proposition 4: With (A3),  $g_1(\alpha y; \alpha y^*) = \alpha y \cdot (1 - f(\alpha y; \alpha y^*)) = \alpha y(1 - f(y; y^*)) = \alpha g_1(y; y^*)$ , i.e.  $g_1$  is homogeneous of degree 1 in  $(y; y^*)$ . Similarly,  $g_2$  is homogeneous of degree 1 in  $(y; y^*)$ . Furthermore  $h_1$ , the inverse function of  $g_1$  inherits the homogeneity of degree 1 in  $(k; y^*)$ , since  $g_1(\alpha y; \alpha y^*) = \alpha g_1(y; y^*) = \alpha k$  implies  $h_1(\alpha k; \alpha y^*) = \alpha y = \alpha h_1(k; y^*)$ . Similarly, it can be demonstrated that  $h_2$ , the inverse of  $g_2$ , is homogeneous of degree 1 in  $(k; y^*)$ .

Therefore, by Euler's theorem,

$$0 = \frac{\partial^2 h_1}{\partial k \partial y^*} \cdot y^* + \frac{\partial^2 h_1}{\partial k^2} \cdot k,$$

With the strict concavity of  $h_1$ , we then have

$$\frac{\partial^2 h_1}{\partial k \partial y^*} = -\frac{\partial^2 h_1}{\partial k^2} \cdot \frac{k}{y^*} > 0. \quad (11)$$

Similarly, with the strict convexity of  $h_2$ ,

$$\frac{\partial^2 h_2}{\partial k \partial y^*} = -\frac{\partial^2 h_2}{\partial k^2} \cdot \frac{k}{y^*} < 0. \quad (12)$$

Now  $\lambda(k; y^*) = h_1(k; y^*) - h_2(k; y^*)$  with  $\lambda(0; y^*) = 0$ . Clearly,  $h_1$  and  $h_2$  are  $C^2$  so that (11) and (12) imply that  $\frac{\partial^2}{\partial y^* \partial k} \lambda(k; y^*) = \frac{\partial^2}{\partial k \partial y^*} \lambda(k; y^*) > 0$ ; hence (I).

From  $\lambda(0; y^*) \equiv 0$  follows the identity  $\frac{\partial}{\partial y^*} \lambda(0; y^*) \equiv 0$ , which together with

$\frac{\partial^2}{\partial k \partial y^*} \lambda(k; y^*) > 0$  yields  $\frac{\partial}{\partial y^*} \lambda(k; y^*) > 0$  for all  $k > 0, y^* > 0$ . Therefore (II).

Finally,  $\frac{\partial}{\partial k} \lambda(\tilde{k}(y^*); y^*) = 0$  together with  $\frac{\partial^2}{\partial y^* \partial k} \lambda(k; y^*) > 0$  and strict concavity

of  $\lambda$  in  $k$  implies that  $\tilde{k}(y^*) < \tilde{k}(y^{**})$  for  $0 < y^* < y^{**}$ , i.e. (III). ■■

Proof of Proposition 5. We divide the proof into three parts:

(i) From Lemma 3 and its proof, we know that for any  $y^* > 0$ , there exists a unique  $\tilde{k}(y^*) \in (0, \bar{y}(y^*)/2)$  such that  $\lambda(n; y^*) < \lambda(m; y^*)$  for

$$0 \leq n \leq m \leq \tilde{k}(y^*).$$

(ii) For  $0 < y^* < y^{**}$ , set  $k(y^*, y^{**}) \equiv \tilde{k}(1) \cdot y^*$ . Then  $0 < k < k(y^*, y^{**})$  implies

$$0 < k/y^{**} < k/y^* < \tilde{k}(1). \text{ Hence by (i), } \lambda(k/y^{**}; 1) < \lambda(k/y^*; 1).$$

- (iii) Let  $0 < y^* < y^{**}$  and  $0 < k < k(y^*, y^{**})$ . Then by Lemma 4 and (ii),  
 $\lambda(k; y^{**})/y^{**} = \lambda(k/y^{**}; 1) < \lambda(k/y^*; 1) = \lambda(k; y^*)/y^*$ . ■■

Proof of Lemma 5:

By Corollary 1, when  $k \in (\frac{\bar{y}}{2}, \hat{y}f(\hat{y})]$ , only complete-profit-transfer Take-it equilibria yield positive payoffs for PH. Then

$$\begin{aligned} E\pi_p &= \frac{1}{y^*} \int_{y_l}^{y_m} (a-c)y dy = \frac{1}{2y^*} (a-c)(y_m^2 - y_l^2). \text{ Moreover,} \\ \frac{\partial E\pi_p}{\partial k} &= \frac{1}{y^*} (a-c) \left( y_m \frac{\partial y_m}{\partial k} - y_l \frac{\partial y_l}{\partial k} \right) \\ &= \frac{1}{y^*} (a-c) \left( \frac{y_m}{g'_2(y_m)} - \frac{y_l}{g'_2(y_l)} \right) < 0 \end{aligned}$$

given that  $y_m \geq \hat{y} \geq \bar{y} \geq y_l$  and other properties of  $g_2(y)$ , i.e.,  $g'_2(y_m) < 0$  and  $g'_2(y_l) > 0$ . ■■

Proof of Lemma 6: When  $k \in \left[0, \frac{\bar{y}}{2}\right]$ , only complete-profit-transfer and incomplete-profit-transfer Take-it equilibria yield positive payoffs for PH. Thus

$$\begin{aligned} E\pi_p &= \frac{1}{y^*} \left\{ \int_{y_l}^{y_m} (a-c)y dy + \int_{y_m}^{y_r} [(a-c)yf(y) + L] dy \right\} \\ &= \frac{1}{y^*} \left\{ \frac{1}{2} (a-c)(y_m^2 - y_l^2) + (a-c) \left[ \int_{y_m}^{y_r} yf(y) dy + k(y_r - y_m) \right] \right\} \end{aligned}$$

It suffices to show  $\frac{1}{2}(a-c)(y_m^2 - y_l^2)$  and  $(a-c) \left[ \int_{y_m}^{y_r} yf(y) dy + k(y_r - y_m) \right]$  are both decreasing in  $k$  to prove the assertion.

Denote

$$D(k) \equiv \frac{\partial(y_m^2 - y_l^2)}{\partial k} = 2 \left( y_m \frac{\partial y_m}{\partial k} - y_l \frac{\partial y_l}{\partial k} \right) = 2 \left( \frac{y_m}{g'_2(y_m)} - \frac{y_l}{g'_2(y_l)} \right).$$

By the convexity of  $g_1$  and concavity of  $g_2$  we can infer  $g''_1(y_m) > 0, g''_2(y_l) < 0$  and  $\frac{g'_1(y_m)}{[g'_1(y_m)]^3} - \frac{g'_2(y_l)}{[g'_2(y_l)]^3} < 0$ . Thus it is clear that  $D(k)$  is decreasing in  $k \in \left[0, \frac{\bar{y}}{2}\right]$ :

$$D'(k) = \frac{g'_1(y_m) - y_m g''_1(y_m)}{[g'_1(y_m)]^3} - \frac{g'_2(y_l) - y_l g''_2(y_l)}{[g'_2(y_l)]^3} < 0$$

Recall that  $g'_1[y_m(\tilde{k})] = g'_2[y_l(\tilde{k})]$  at  $\tilde{k}$ , hence

$$D(\tilde{k}) = \frac{2}{g'_1(y_m)}(y_m - y_l) > 0.$$

By Lemma 3 we know that  $g'_1(\bar{y}) \geq 1$  and  $g'_2(\bar{y}) = 0$  so that

$$D\left(\frac{\bar{y}}{2}\right) = 2\bar{y} \cdot \left(\frac{1}{g'_1(\bar{y})} - \frac{1}{g'_2(\bar{y})}\right) < 0$$

Thus by the intermediate value theorem, there exists a  $\overset{\circ}{k} \in \left(\tilde{k}, \frac{\bar{y}}{2}\right)$  such that

$D(\overset{\circ}{k}) = 0$  and  $D(k) \leq 0$  for  $k \in \left(\overset{\circ}{k}, \frac{\bar{y}}{2}\right)$ . It is then an immediate consequence that

$\frac{1}{2}(a-c)(y_m^2 - y_l^2)$  is decreasing in  $\left[\overset{\circ}{k}, \frac{\bar{y}}{2}\right]$ , since its first order derivative is non-positive in the relevant range.

Now we verify that the second term,  $(a-c)\left[\int_{y_m}^{y_r} yf(y)dy + k(y_r - y_m)\right]$  is decreasing

in  $k$ . Observe that

$$\frac{\partial E\pi_p}{\partial k} = \frac{1}{y^*} \left\{ \frac{1}{2}(a-c)D(k) + (a-c) \left[ \frac{g_2(y_r)}{g'_2(y_r)} - \frac{g_2(y_m)}{g'_1(y_m)} + k \left[ \frac{1}{g'_2(y_r)} - \frac{1}{g'_1(y_m)} \right] \right] \right\} .$$

??

It is easy to verify that both  $\frac{g_2(y_r)}{g'_2(y_r)} - \frac{g_2(y_m)}{g'_1(y_m)}$  and  $\left[\frac{1}{g'_2(y_r)} - \frac{1}{g'_1(y_m)}\right]$  are both negative due to the convexity (concavity) of  $g_1, g_2$  for  $k$  in the relevant range.

This, together with the term  $\frac{1}{2}(a-c)(y_m^2 - y_l^2)$  decreasing, implies that there exist

$\overset{\circ}{k} \in \left(0, \frac{\bar{y}}{2}\right)$  such that  $E\pi_p(k)$  is decreasing in  $\left[\overset{\circ}{k}, \frac{\bar{y}}{2}\right]$ . ■■

### Proof of Lemma 7:

Denote

$$E(k; y^*) \equiv \frac{E\pi_p(k; y^*)}{E\pi(k; y^*)}, \text{ then}$$

$$E(k; y^*) = \frac{\frac{1}{y^*} \left\{ \int_{y_l}^{y_m} (a-c)ydy + \int_{y_m}^{y_r} [(a-c)yf(y; y^*) + L]dy \right\}}{\frac{1}{y^*} \int_0^{y^*} (a-c)ydy}$$

$$= \frac{\frac{1}{2}(a-c)(y_m^2 - y_l^2) + (a-c) \left[ \int_{y_m}^{y_r} yf(y; y^*)dy + k(y_r - y_m) \right]}{\frac{1}{2}(a-c)(y^*)^2}$$

$$= \frac{(y_m^2 - y_l^2) + 2\left[\int_{y_m}^{y_r} yf(y; y^*)dy + k(y_r - y_m)\right]}{(y^*)^2}$$

Let  $\lambda > 0$ . Then by the homogeneity of  $y_l$ ,  $y_m$ ,  $y_r$  and  $f(y; y^*)$  from Lemma 4, we know that

$$\begin{aligned} E(\lambda k; \lambda y^*) &= \frac{\lambda^2 (y_m^2 - y_l^2) + 2\left[\int_{\lambda y_m}^{\lambda y_r} yf(y; \lambda y^*)dy + \lambda^2 k(y_r - y_m)\right]}{\lambda^2 (y^*)^2} \\ &= \frac{(y_m^2 - y_l^2) + 2\left[\int_{y_m}^{y_r} yf(y; y^*)dy + k(y_r - y_m)\right]}{(y^*)^2} \\ &= E(k; y^*) \quad \blacksquare \blacksquare \end{aligned}$$

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