Bilateral Consistency and Converse Consistency in Axiomatizations and Implementation of the Nucleolus

Min-Hung Tsay

Institute of Economics, Academia Sinica, Taiwan
(with Cheng-Cheng Hu and Chun-Hsien Yeh)

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Motivation

The nucleolus is a central solution concept in the theory of TU games and has been applied to many resource allocation problems.

- What are the essential differences between the nucleolus and other solution?
- We examine this issue from two aspects: axiomatic and non-cooperative.
- We show that bilateral consistency and converse consistency principles play key roles in this regard.

Motivation

- **Bilateral Consistency:** a rule is bilaterally consistent if an alternative is chosen by the rule for a problem, then for the associated "two-agent reduced problem" obtained by imagining the departure of all other agents with their "components of the alternative", and reassessing the options open to the two-agent subgroup, it chooses the restriction of the alternative to that subgroup.
- **Converse Consistency:** a rule is conversely consistent if an alternative is such that for each problem and each proper two-agent subgroup, the rule chooses the restriction of the alternative to this subgroup for the reduced problem it faces, then the alternative should be chosen for the initial problem by the rule.

Motivation

Bilateral consistency and converse consistency principles have been applied to and extensively studied for many models such as

- Transferable Utility (TU) games and Non-TU games
- Arrovian social choice problems
- Classical exchange economy
- Nash bargaining problems
- Matching problems
- Bankruptcy problems
- *etc.*

For a survey, see Thomson (2004).

Motivation

Maschler and Owen (1989) points out that in TU games, consistency principle (a stronger version of bilateral consistency) can be used to "distinguish" solutions from axiomatic perspective. To see this, let Pareto optimality, single-valuedness, equal treatment property (or anonymity), and covariance be the basic properties.

Motivation

- Hart and Mas-Colell (1989): ψ: the basic properties + self-consistency $\Leftrightarrow$ ψ is the Shapley value.
- Solobev (1975): ψ: the basic properties + max-consistency $\Leftrightarrow$ ψ is the prenucleolus.
- Moulin (1985): ψ: the basic properties + complement-consistency $\Leftrightarrow$ ψ is the Equal Allocation of Non-Separable Cost (EANSC) value.

Motivation

Chang and Hu (2007) show the followings.

- ψ: the basic properties + bilateral self-consistency (or converse self-consistency) $\Leftrightarrow$ ψ is the Shapley value.
- ψ: the basic properties + bilateral complement-consistency (or converse complement-consistency) $\Leftrightarrow$ ψ is the EANSC value.

Namely, the Shapley value and the EANSC value can be distinguished by different bilateral consistency and converse consistency properties in TU games.

Motivation

Our paper offer positive answers to the following questions.

1. Whether such differentiation between the nucleolus and other solution can be obtained in other allocation problems.

As suggested by Krishna and Serrano (1996), the properties of a solution can be used as guides in designing a non-cooperative game that implements the solution.

2. Whether a non-cooperative justification for the nucleolus can be obtained by revising a game, based on bilateral consistency and converse consistency of the nucleolus, that implements other solution and exploits different bilateral consistency and converse consistency properties.

Applications

To serve our purpose, we consider a class of cost allocation problems, which exemplifies the following problems.

Applications

Taxi-Fee Sharing Problem

- Several agents are jointly riding a taxi, different agents having different destinations and different uses for it.
- The further the destination an agent has, the longer the distance the agent needs.
- The taxi that accommodates a given agent with a certain distance accommodates any nearer distance that any agent has.

How should the tax-fee be shared among them?

Applications

Airport Problem

- Several airlines are jointly using an airstrip, different airlines having different needs for it.
- The larger the planes an airline flies, the longer the airstrip it needs.
- An airstrip that accommodates a given airplane accommodates any smaller airplane any airline flies.

How should the maintenance cost of the airstrip be shared among the airlines?

Applications

Irrigation Problem

- Ranchers are distributed along an irrigation ditch.
- The rancher closest to the headgate only needs that the segment from the headgate to his field, the "first segment", be maintained, the second closest rancher needs that the first two segments be maintained, and so on.
- The cost of maintaining a segment used by several agents is incurred only once, independently of how many agents use it.

How should the maintenance cost of the ditch be shared among the ranchers?

Applications

Compensation Problem

- An object is to be allocated to one of several agents, each agent being characterized by the value of he assigns to it.
- All agents have equal rights on the object.
- Efficiency requires that the object should be assigned to the agent who values it the most.

How should those agents, who do not receive the object, be compensated?

The model: formal definition

- This class of cost sharing problems has been studying for many years. One famous example is the study of 25 irrigation ditches located in south-central Montana, USA.
- However, there was no formal discussion and rigorous analysis about this class of cost sharing problems until Littlechild and Owen (1973).
- How do they formulate the class of the problems?

The model: formal definition

$$\varphi\left(N \equiv \{1,2,3\}, c \equiv (c_1, c_2, c_3) \in \mathbb{R}^N_+\right)$$

$$= (x_1, x_2, x_3) \in \mathbb{R}^N \text{s.t. for each } i \in N, 0 \leq x_i \leq c_i \text{ and } \sum_{i \in N} x_i = \max_{j \in N} c_j.$$

- The property, $0 \leq x_i \leq c_i$, is referred to as reasonableness and says that agent i should not receive a subsidy and should not contribute more than his cost parameter (the stand-alone cost).
- The property, $\sum_{i \in N} x_i = \max_{j \in N} c_j$, is referred to as efficiency and says that a rule should collect the exact amount of money to complete the work.

The model: geometric representation and useful notations

Geometry representation of the airport problem To facilitate my presentation, let $N \equiv \{1, \cdots, n\}$ and $c_1 \leq \cdots \leq c_n$. We call the differences c_1, $c_2 - c_1$, $c_3 - c_2$, and so on, segmental costs. c_1 is the cost of segment 1, $c_2 - c_1$ is the cost of segment 2, $c_3 - c_2$ is the cost of segment 3.

Examples of rules: the constrained equal benefits rule

The first rule is defined as follows:

- This rule focuses on how much agents can benefit from joining the cost sharing project. It equalizes agents' benefits subject to no one contributing a negative amount.

Examples of rules: the constrained equal benefits rule

Formally,

Constrained Equal Benefits rule, CEB: For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$CEB_i(N, c) \equiv \max\{c_i - \beta, 0\},$$

where $\beta \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEB_i(N, c) = \max_{j \in N} c_j$.
Geometric explanation of the CEB rule

# Examples of rules: the constrained equal contributions rule

The next rule can be calculated by the following algorithm. Start by requiring that all agents in N should contribute equally until there are $\lambda^1 \in \mathbb{R}_+$ and a group of agents $\{1, \cdots, l^1\}$ (if there are several such groups, pick the group containing the agent with the largest index) such that $\lambda^1 l^1 = c_{l^1}$. Each agent in $\{1, \cdots, l^1\}$ then contributes λ^1. The algorithm next requires that all agents in $\{l^1 + 1, \cdots, n\}$ should contribute equally until there are $\lambda^2 \in \mathbb{R}_+$ and a group of agents $\{l^1 + 1, \cdots, l^2\}$ (if there are several such groups, pick the group containing the agent with the largest index) such that $\lambda^2 (l^2 - l^1) = c_{l^2} - c_{l^1}$. Each agent in $\{l^1, \cdots, l^2\}$ then contributes λ^2.

# Examples of rules: the constrained equal contributions rule

Continue this process until the total cost c_n is covered. This algorithm amounts to the following formula.

Constrained Equal Contributions rule, CEC: For each $(N, c) \in \mathcal{A}$,

$$
\begin{aligned}
CEC_1(N,c) \equiv & \quad \min_{1 \le k \le n} \left\{ \frac{c_k}{k} \right\} \\
CEC_i(N,c) \equiv & \quad \min_{i \le k \le n} \left\{ \frac{c_k - \sum_{p=1}^{i-1} CEC_p(N,c)}{k-i+1} \right\}, \quad \text{where } 2 \le i \le n
\end{aligned}
$$

Examples of rules: the nucleolus

Our next rule is central to our paper and a direct application of the nucleolus in TU games. For general games, the payoff vector chosen by the nucleolus is difficult to compute since it involves a sequence of linear programs. However, for the airport problem, its contributions vector can be obtained by the following algorithm, which shares a similar spirit with the one for the CEC rule. The algorithm starts by requiring that all agents in $N \backslash \{n\}$ should contribute equally until there are $\beta^1 \in \mathbb{R}_+$ and a group of agents $\{1, \cdots, p^1\}$ (if there are several such groups, pick the group containing the agent with the largest index) together with the last agent such that $\beta^1 (p^1 + 1) = c_{p^1}$. Each agent in $\{1, \cdots, p^1\}$ then contributes β^1.

Examples of rules: the nucleolus

The algorithm next requires that all agents in $\{p^1+1,\cdots,n-1\}$ should contribute equally until there are $\beta^2 \in \mathbb{R}_+$ and a group of agents $\{p^1+1,\cdots,p^2\}$ (if there are several such groups, pick the group containing the agent with the largest index) together with the last agent such that $\beta^2\left(p^2-p^1+1\right)=c_{p^2}-\beta^1 p^1$. Each agent in $\{p^1,\cdots,p^2\}$ then contributes β^2. Continue this process until c_n is covered. The algorithm amounts to Sönmez (1994)'s formula defined next.

Nucleolus, Nu: For each $(N,c)\in\mathcal{A}$,

$$\begin{aligned} Nu_1(N,c) &\equiv \min_{1\le k\le n-1}\left\{\frac{c_k}{k+1}\right\} \\ Nu_i(N,c) &\equiv \min_{i\le k\le n-1}\left\{\frac{c_k-\sum_{p=1}^{i-1}Nu_p(N,c)}{k-i+2}\right\}, \quad \text{where } 2\le i\le n-1 \\ Nu_n(N,c) &\equiv c_n-\sum_{p=1}^{n-1}Nu_p\left(N,c\right). \end{aligned}$$

Basic properties

We consider the following properties. The first one says that if the costs of two agents are equal, they should contribute equal amounts.

Equal treatment of equals: For each $(N, c) \in \mathcal{A}$ and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $\varphi_i(N, c) = \varphi_j(N, c)$.

Suppose that the cost of the last agent increases by some amount. Since this addition to the runway is only used by this agent, the second property requires that she should be responsible for all of this incremental cost.

Last-agent cost additivity: For each pair $\{(N, c), (N, c')\}$ of elements of $\mathcal{A}$ and each $\delta \in \mathbb{R}_+$, if $c'_n = c_n + \delta$ and for each $j \in N \backslash \{n\}$, $c'_j = c_j$, then $\varphi_n(N, c') = \varphi_n(N, c) + \delta$.

Bilateral consistency and converse consistency

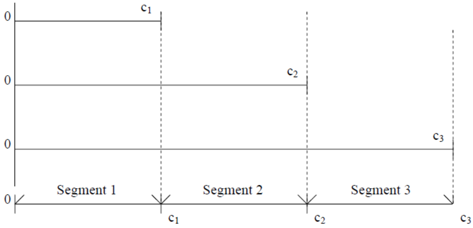
Our next properties have to do with variable population and re-evaluate the situation from the viewpoints of the remaining agents (namely, derive a reduced problem). In contrast to other models of fair allocation, for which a unique formulation of a reduced problem usually stands out as the most natural, there are many formulations of a reduced problem in the airport problem. Informally, this is because what has to be divided is not a homogeneous whole, but it is composed of segments used differently by different agents.

Bilateral consistency and converse consistency

Potters and Sudhölter (1999) propose two formulations.

- Left-endpoint Subtraction (LS) formulation.
- Right-endpoint Subtraction (RS) formulation.

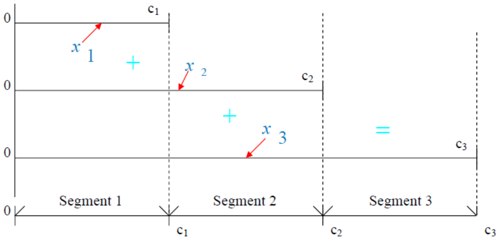
Bilateral consistency and converse consistency



There are three airlines. For each airline, say airline i, the cost of satisfying its need is represented by the cost parameter c_i. For simplicity, assume that $c_1 < c_2 < c_3$. Thus, airline 1 uses segment 1. Airline 2 uses segments 1 and 2. Airline 3 uses segments 1, 2 and 3. The cost of segment 1 is c_1, the cost of segment 2 is $c_2 - c_1$, the cost of segment 3 is $c_3 - c_2$. The total cost of the airstrip is c_3, which is the sum of c_1 (the cost of segment 1), $c_2 - c_1$ (the cost of segment 2), and $c_3 - c_2$ (the cost of segment 3).

Figure : The three-airline problem

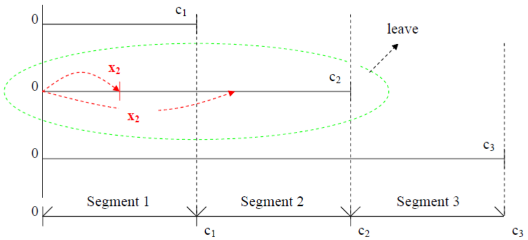
Bilateral consistency and converse consistency



$$(x_1, x_2, x_3) = \varphi\left(\{1,2,3\}; c_1, c_2, c_3\right)$$

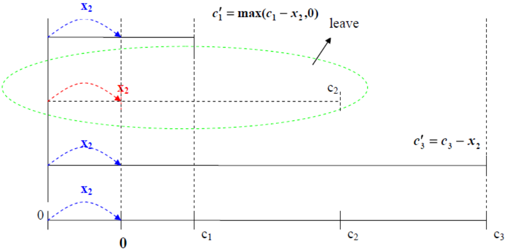
Bilateral consistency and converse consistency

LS formulation:



Bilateral consistency and converse consistency

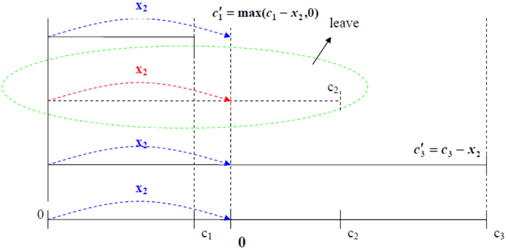
LS consistency:



$$x_1 = \varphi_1(\{1,3\};c_1',c_3') \text{ and } x_3 = \varphi_3(\{1,3\};c_1',c_3')$$

Bilateral consistency and converse consistency

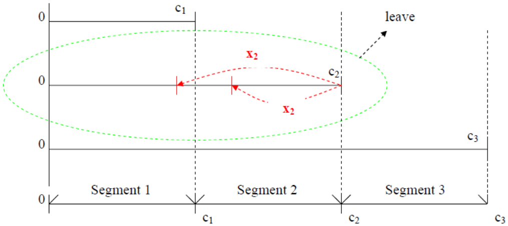
LS consistency:



$$x_1 = \varphi_1(\{1,3\}; c_1', c_3') \text{ and } x_3 = \varphi_3(\{1,3\}; c_1', c_3')$$

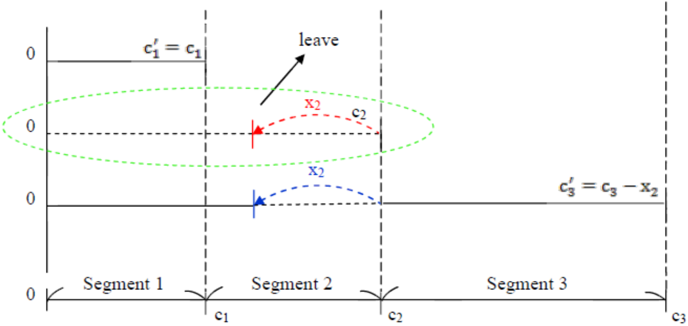
Bilateral consistency and converse consistency

RS formulation:



Bilateral consistency and converse consistency

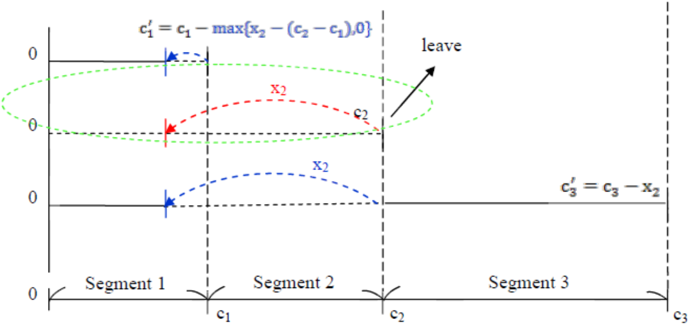
RS consistency:



$$x_1 = \varphi_1\left(\{1,3\}; c'_1, c'_3\right) \text{ and } x_3 = \varphi_3\left(\{1,3\}; c'_1, c'_3\right)$$

Bilateral consistency and converse consistency

RS consistency:



$$x_1 = \varphi_1\left(\{1,3\}; c'_1, c'_3\right) \text{ and } x_3 = \varphi_3\left(\{1,3\}; c'_1, c'_3\right)$$

Bilateral consistency and converse consistency

- Hu et al. (2012) adopt LS formulation to propose the **reduced problem of** (N, c) **with respect to** $N' = \{i, n\}$ **and** x**,** $(N', r^x_{N'})$ to be defined by setting

$$(r^x_{N'})_i \equiv \max\left\{c_i - \sum\nolimits_{k\neq i,n} x_k, 0\right\} \text{ and}$$
$$(r^x_{N'})_n \equiv c_n - \sum\nolimits_{k\neq i,n} x_k.$$

  - **LS bilateral consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| \geq 2$ and each $i \in N \setminus \{n\}$, if $x = \varphi(N, c)$, then $(\{i, n\}, r^x_{\{i,n\}}) \in \mathcal{A}$ and $x_{\{i,n\}} = \varphi(\{i, n\}, r^x_{\{i,n\}})$.
  - **LS converse consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| > 2$ and each $x \in X(N, c)$, if for each $N' \subset N$ with $|N'| = 2$ and $n \in N'$, $x_{N'} = \varphi(N', r^x_{N'})$, then $x = \varphi(N, c)$.

Bilateral consistency and converse consistency

- We use RS formulation to propose the **reduced problem of (N, c) with respect to $N' \equiv \{i, n\}$ and** x, $(N', r^x_{N'})$, is defined by setting

$$(r^x_{N'})_i \equiv \max\left\{\min_{i\leq k, k\neq n}\left\{c_k - \sum_{m\leq k, m\neq i,n} x_m\right\}, 0\right\} \text{ and}$$

$$(r^x_{N'})_n \equiv \max\left\{\min_{n\leq k, k\neq i}\left\{c_k - \sum_{m\leq k, m\neq i,n} x_m\right\}, 0\right\} = c_n - \sum_{m\neq i,n} x_m.$$

Bilateral consistency and converse consistency

- **RS bilateral consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| \geq 2$ and each $i \in N \setminus \{n\}$, if $x = \varphi(N, c)$, then $(\{i, n\}, r^x_{\{i,n\}}) \in \mathcal{A}$ and $x_{\{i,n\}} = \varphi(\{i, n\}, r^x_{\{i,n\}})$.
- **RS converse consistency:** For each $(N, c) \in \mathcal{A}$ with $|N| > 2$ and each $x \in X(N, c)$, if for each $N' \subset N$ with $|N'| = 2$ and $n \in N'$, $x_{N'} = \varphi(N', r^x_{N'})$, then $x = \varphi(N, c)$.

Axiomatizations

Hu et al. (2012) show the following:

Theorem: φ: equal treatments of equals + last-agent cost additivity + LS bilateral consistency (or LS converse consistency) $\Leftrightarrow \varphi = CEB$.

We show the following:

Theorem: φ: equal treatments of equals + last-agent cost additivity + RS bilateral consistency (or RS converse consistency) $\Leftrightarrow \varphi = Nu$.

Axiomatizations

Namely, our axiomatizations and Hu et al. (2012)'s results show that the nucleolus and the CEB rule can be distinguished by different bilateral consistency and converse consistency properties in the airport problem from axiomatic perspective.

The game form

We next ask whether such differentiation can be obtained from non-cooperative perspective.

- To answer this question, we consider Hu et al. (2012)'s game that implements the CEB rule and exploits LS bilateral consistency and LS converse consistency, and revise this game based on RS bilateral consistency and RS converse consistency.
- Let's first introduce Hu et al. (2012)'s game:
  Hu et al. (2012)'s game form.

The game form

We explain Hu et al. (2012)'s game in details.

- In Stage 1, each agent $k \in N \backslash \{n\}$, except for the responder (the last agent), proposes her own contribution $0 \leq x_k \leq c_k$ to the total cost. Due to the exclusion of agent n's leaving in the definitions of LS bilateral consistency and LS converse consistency, agent n plays no role in Stage 1. In Stage 2, the responder either accepts or rejects to contribute the residual cost x_n.

The game form

- The contributions of agents i and n are specified based on the reduced problem underlying LS bilateral consistency and LS converse consistency as follows. Imaging that there is a fair coin to select one of the two agents. The chosen agent, say agent $l \in \{i, n\}$, has no choice but pick the group $N \backslash \{i, n\}$ and use the sum of their contributions $\sum_{k \neq i,n} x_k$ to cover her cost c_l. Namely, her contribution is $\max \left\{0, c_l - \sum_{k \neq i,n} x_k \right\}$. The other agent contributes the remainder $x_i + x_n - \max \left\{0, c_l - \sum_{k \neq i,n} x_k \right\}$.

The game form

- We construct the following game form: The game form.

The game form

In contrast to Hu et al. (2012)'s game, we introduce an additional stage. This addition is necessary since due to the essence of the reduced problem underlying RS bilateral consistency and RS converse consistency, the agent chosen by agent n in Stage 2 is entitled to minimize her contribution by choosing a group of agents and using this group's contributions to cover the cost of building the part of the runway she and all agents in the group can use.

The game form

In contrast to Hu et al. (2012)'s game, we specify the contributions of agents i and n based on the reduced problem underlying RS bilateral consistency and RS converse consistency as follows. Imagine that there is a fair coin to select one of the two agents. The chosen one, say agent $l \in \{i, n\}$, picks a group, say S, from $2^{N \setminus \{i,n\}}$ and takes $\sum_{k \in S} x_k$ to cover $\max_{k \in S}\{c_l, c_k\}$ (the cost of building the part of the runway agent l and all agents in S can use). Namely, her contribution is $\max\left\{\max_{k \in S}\{c_l, c_k\} - \sum_{k \in S} x_k, 0\right\}$. The other agent contributes the remainder $x_i + x_n - \max\left\{\max_{k \in S}\{c_l, c_k\} - \sum_{k \in S} x_k, 0\right\}$.

The game form

In Hu et al. (2012)'s game, agent l has no choice but to pick the group $N\backslash\{i,n\}$ and takes the sum of their contributions $\sum_{k\in N\backslash\{i,n\}} x_k$ to cover her cost c_l (the cost of building the part of the runway agent l can use). Namely, she contributes $\max\left\{0, c_l - \sum_{k\in N\backslash\{i,n\}} x_k\right\}$. In our game, agent l picks the group S from $2^{N\backslash\{i,n\}}$ and takes the sum of their contributions $\sum_{k\in S} x_k$ to cover $\max_{k\in S}\{c_l, c_k\}$ (the cost of building the part of the runway agent l and all agents in S can use). Namely, her contribution is $\max\left\{0, \max_{k\in S}\{c_l, c_k\} - \sum_{k\in S} x_k\right\}$.

Implementation

Hu et al. (2012) show the following:

Theorem: (Existence result) There exists a subgame perfect equilibrium of $\Gamma_{CEB}(N, c)$ with outcome $CEB(N, c)$.

Theorem: (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_{CEB}(N, c)$ is $CEB(N, c)$.

We show the following:

Theorem: (Existence result) There exists a subgame perfect equilibrium of $\Gamma_{Nu}(N, c)$ with outcome $Nu\,(N, c)$.

Theorem: (Uniqueness result) Each subgame perfect equilibrium outcome of the game $\Gamma_{Nu}(N, c)$ is $Nu\,(N, c)$.

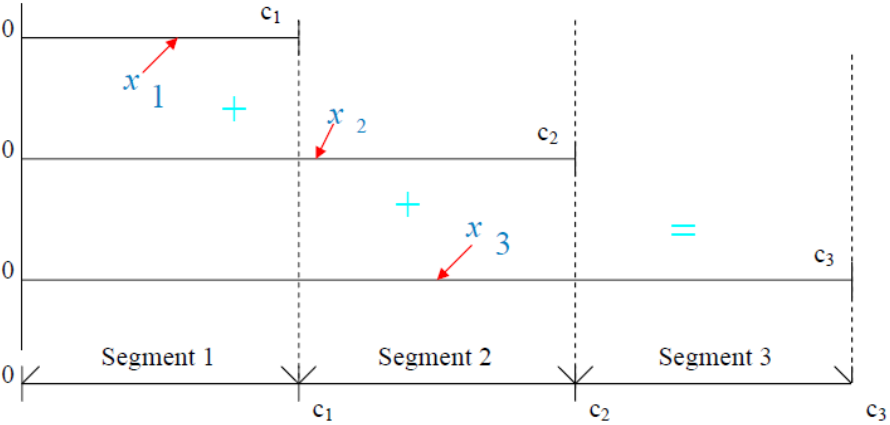
Implementation

In the literature on Nash program, it is typical to introduce a game to implement a solution or a class of solutions. However, there was no attempt to construct a game that does not only implement a solution but also shed light into fundamental differences between the solution and others. Our game does not only implement the nucleolus but also illustrate the key roles bilateral consistency and converse consistency principles play in differentiation between the nucleolus and the CEB rule. To our best knowledge, our paper is the first paper to observe such interesting phenomenon from non-cooperative perspective.

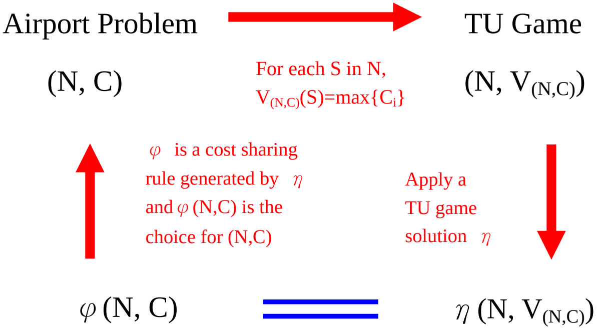
Concluding remarks

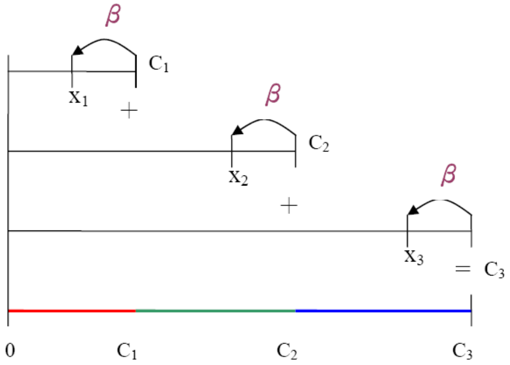
- Our paper points out that bilateral consistency and converse consistency principles deepen our understanding of fundamental differences between the nucleolus and the CEB rule from axiomatic as well as strategic perspectives.
- It can be shown that the CEC rule is RS bilaterally consistent but not RS conversely consistent. Thus, an implementation of the CEC rule cannot be obtained by revising the expected contributions of agents i and n in Hu et al. (2012)'s game based on RS bilateral consistency and RS converse consistency. This fact shows that converse consistency principle plays an important role.
- It would be interesting to investigate whether such phenomena occur between other rules in other allocation problems.

Thank you!!



$$(x_1, x_2, x_3) = \varphi\left(\{1,2,3\}; c_1, c_2, c_3\right)$$


Airport Problem
(N, C)
For each S in N,
$V_{(N,C)}(S)=\max\{C_i\}$
TU Game
$(N, V_{(N,C)})$
φ is a cost sharing rule generated by η and φ (N,C) is the choice for (N,C)
Apply a TU game solution η
φ (N, C)
$\eta\ (N, V_{(N,C)})$

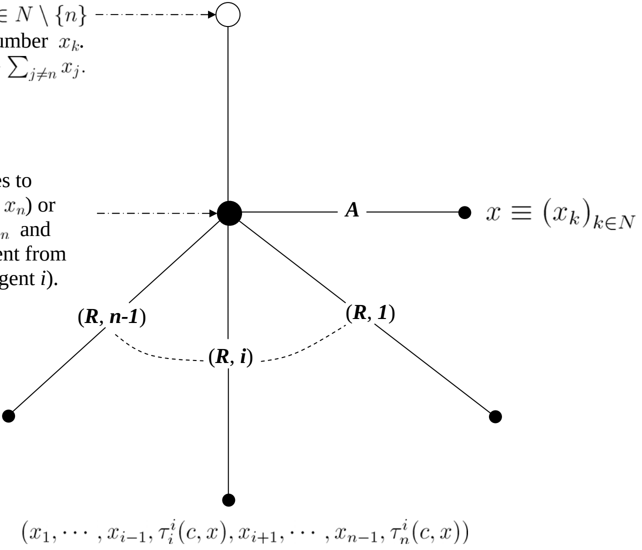

β
C1
x1
+
β
C2
x2
+
β
x3
= C3
0
C1
C2
C3

Stage 1:

Each agent $k \in N \setminus \{n\}$ announces a number x_k.
Let $x_n \equiv c_n - \sum_{j \neq n} x_j$.

Stage 2:

Agent *n* decides to take ***A*** (accept x_n) or (***R*, *i***) (reject x_n and choose one agent from $N \setminus \{n\}$, say agent *i*).



Stage 1:
Each agent $k \in N \setminus \{n\}$ announces a number $x_k \in \mathbb{R}$ such that $0 \leq x_k \leq c_k$. Let $x_n = c_n - \sum_{h \neq n} x_h$.

Stage 2:
Agent *n* decides to take ***A*** (accept x_n) or (***R, i***) (reject x_n and choose one agent from $N \setminus \{n\}$, say agent *i*).

Stage 3:
A fair coin selects one agent from $\{i, n\}$, say agent *l*. Agent *l* picks a group of agents from $2^{N \setminus \{i,n\}}$, say ***S***. Let $h \in \{i, n\} \setminus \{l\}$.