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On the duality between prior beliefs and trading demands ☆

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Abstract

In an abstract model with asymmetric information, we show that there is a duality relationship between the prior beliefs and trading demands of bets for any given individual. Then we aggregate all the agents to obtain a second duality relationship between common prior beliefs and trading possibilities. We easily derive from these relationships the no trade theorem and its converse. General efficiency results can be obtained. Moreover, our framework is sufficiently general to cover special cases proved previously (for example, Econometrica 62 (1994) 1327; Discussion Paper 83, Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, 1995; J. Econom. Theory 91 (2000) 127; Games Econom. Behav. 24 (1998) 172. Yet, our arguments are both simple and intuitive.

JEL classification: C70; D82; D84

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1. Introduction

If agents share the same prior beliefs, they will not trade for purely informational reasons, even in the presence of asymmetric information. Depending on the trading

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environment, we can find several versions of this "no trade theorem" in the literature,[1] where the common prior assumption is usually adopted. An extensive survey of the common prior assumption and its relation to the no trade literature can be found in [9].

In recent years, Morris [8], Feinberg [3,4] and Samet [12] have essentially shown that the converse of the no trade theorem is also true. The main purpose of this paper is to give a simple and intuitive proof that generalizes most of the existing results in the form of a no trade principle, which states the equivalence between the common prior assumption and the no trade conclusion.

It turns out that, in many applications, only the betting case is relevant. We begin with a non-empty set of agents. Without loss of generality, we assume in the main body of the paper that they are all risk-neutral. Then, we consider an uncertainty environment described by a set Ω, which consists of all possible states of the world. A prior of an agent $i \in I$ is described by a probability distribution over Ω. Betting may occur among the agents and the state of the world is realized.

We assume that each agent may have many possible prior beliefs. For any given agent, we say that a bet is *positive* if it leads to positive expected gains with respect to all his possible priors. It is common knowledge that each agent's *trading demand set* consists of all his positive bets.

The arguments put forward in this paper can be divided into two major steps. Firstly, we demonstrate that, for any given agent, there exists a duality relationship between the set of all his possible prior beliefs and his trading demand set. This result corresponds to the well-known economic intuition that different opinions would give rise to different trading demands, and that different trading demands should have come from different opinions. Secondly, by using this duality result at the individual level, we obtain, by a simple aggregation process, the no trade principle as an aggregate duality result in a very general environment.

Some special cases of the no trade principle have been obtained in the economics literature by *other arguments* when the number of agents is *finite* [3,4,8,12].[2]

In his pioneer paper, Morris [8, Lemma A2(iv)] uses the Farkas' Lemma to prove a version of the no trade principle for the special case in which Ω is finite.

Before we proceed with our discussion on Feinberg [3,4] and Samet [12], we need to focus on a more special environment. To do this, we introduce an interim stage, in which agents hold private information, into the model and assume that agents are Bayesian. When an agent obtains his private source of information about the states of the world, he can update his prior beliefs to become posterior beliefs according to Bayes' rule. Betting may then occur and the state of the world will be realized. We are interested in examining the no trade principle from the interim viewpoint,[3] when the private information and posterior beliefs of each agent are all given.

[1] See for example, [7,11,13,15].

[2] For the finite type case, and with a finite number of agents, the no trade principle is quite well known in combinatorial geometry. It is an easy consequence of the theory of separation of convex cones, e.g. [2].

[3] Note that Morris [8] states the interim no trade principle from an ex ante viewpoint, not from an interim viewpoint.

In a working paper, Feinberg [3] proves the interim no trade principle for finite type spaces by directly solving a system of linear homogeneous equations. Attempting to solve an aggregate problem directly can explicitly provide the required bets when there is no common prior. However, if proving the no trade principle were the only concern, his proofs would be unnecessarily long.

Based on the observation that prior beliefs can be generated by posterior beliefs when Ω is finite, Samet [12] is able to provide a short proof of the interim no trade principle for finite type spaces by using the Separation Theorem in the product space $\prod_{i\in I} R^{|\Omega|}$. Notice, however, that he is directly proving a duality theorem in the aggregate.

Using Sion's [14] minimax theorem, Feinberg [4] generalizes the interim no trade principle to a special class of compact type spaces with a finite number of agents. In his paper, Feinberg has to assume that posterior beliefs can only vary in a continuous manner across states.

Based on an apparently more intuitive and simpler argument, this paper shows that the interim no trade principle is true in general for a compact type space with little restriction as to how posterior beliefs should change according to the information structure. As a result, the no trade principle described in this paper is by far the most general one.[4]

The rest of this paper proceeds as follows. We introduce the model in Section 2. After proving the no trade principle (Theorem 1), we summarize the first and second duality relationships that we have discovered. Then we show how the interim no trade principle (Theorem 2) can be obtained as a special case when an information structure is introduced at an interim stage. Concluding remarks are presented in Section 3. Appendix A contains a discussion on how we may generalize the model. Essential proofs can be found in Appendix B.

2. The no trade principle

In this section, we set up an uncertainty environment and characterize a necessary and sufficient condition for the existence of common priors. This condition is very general and the result can be interpreted as a no trade principle.

Let Ω be a compact Hausdorff space and $C(\Omega)$ be the Banach space of all continuous functions on Ω under the sup-norm. Let $\mathscr{P}(\Omega)$ denote the set of all regular probability measures on the Borel σ-algebra $\mathscr{F}$. We regard $\mathscr{P}(\Omega)$ as a subset of $rca(\mathscr{F})$, the space of all signed regular Borel measures of bounded variation. It is well known that $rca(\mathscr{F})$, equipped with the variation norm, is isometrically isomorphic to the norm dual of $C(\Omega)$. Unless otherwise specified, the topology we endow on $\mathscr{P}(\Omega)$ is always the weak* topology. Then $\mathscr{P}(\Omega)$ is a compact space.

Let I denote the (non-empty) index set of all agents. Let $Q = \{K_i\}_{i\in I}$ be a collection of non-empty convex compact subsets of $\mathscr{P}(\Omega)$. Each K_i is interpreted as

[4] See, however, [1], Remarks 2, 3 and 5.

the set of *all possible prior beliefs* of agent i. We say that the agents have a (possible) *common prior* if $\bigcap_{i\in I} K_i \neq \emptyset$.

Recall that a subset X in a linear space is a cone if for each $\lambda > 0$ and $x \in X$, we have $\lambda x \in X$. For any non-empty subset K of $\mathscr{P}(\Omega)$, let us define

$$\mathscr{C}_K = \left\{ f \in C(\Omega): \int_\Omega f \, d\mu > 0 \text{ for all } \mu \in K \right\}.$$

It is clear that $\mathscr{C}_K$ is a convex cone containing all the positive continuous functions but not 0.

Lemma 1. *Let K be a non-empty compact subset of $\mathscr{P}(\Omega)$. Then*

(i) *$\mathscr{C}_K$ is open.*
(ii) *If, in addition, K is convex, then*

$$K = \left\{ \mu \in \mathscr{P}(\Omega): \int_\Omega f \, d\mu > 0 \text{ for all } f \in \mathscr{C}_K \right\}.$$

The following corollary is a direct consequence of Lemma 1(ii).

Corollary 1. *Suppose that for each $i \in I$, K_i is a non-empty compact convex subset of $\mathscr{P}(\Omega)$. Then, $\mu \in \bigcap_{i\in I} K_i$ if and only if $\int_\Omega f \, d\mu > 0$ for all $f \in \bigcup_{i\in I} \mathscr{C}_{K_i}$.*

Remark 1. It is interesting to note that the converse of Lemma 1 is also true. Let $C \subseteq C(\Omega)$ be any open convex cone containing all the positive continuous functions but not 0. We define

$$K_C = \left\{ \mu \in \mathscr{P}(\Omega): \int_\Omega f \, d\mu > 0 \text{ for all } f \in C \right\}.$$

It is clear that K_C is a non-empty convex set. We can also show that K_C is compact and

$$C = \left\{ f \in C(\Omega): \int_\Omega f \, d\mu > 0 \text{ for all } \mu \in K_C \right\}.$$

It follows that the collection of all the compact convex sets in $\mathscr{P}(\Omega)$ is in one–one correspondence with the collection of all the open convex cones, each of which contains all the positive continuous functions but not 0 in $C(\Omega)$.

By Lemma 1(i), for each $i \in I$, $\mathscr{C}_{K_i}$ is an open convex set in $C(\Omega)$. Let us denote by $\sum_{i\in I} C_{K_i}$ the set of all finite sums of elements from $\bigcup_{i\in I} C_{K_i}$. Then, it is easy to see that $\sum_{i\in I} \mathscr{C}_{K_i} = \text{con } (\bigcup_{i\in I} \mathscr{C}_{K_i})$ which is an open convex set in $C(\Omega)$. Thus, $\bigcap_{i\in I} K_i \neq \emptyset$ if and only if there is an $\mu \in \mathscr{P}(\Omega)$ satisfying

$$\int_\Omega f \, d\mu > 0 \quad \text{for all } f \in \sum_{i\in I} \mathscr{C}_{K_i}. \tag{1}$$

Now it is straightforward to prove the following theorem.

Theorem 1 (The no trade principle). *A common prior exists if and only if the zero function on Ω cannot be written as a finite sum of elements from $\bigcup_{i\in I}\mathscr{C}_{K_i}$. That is, $\bigcap_{i\in I}K_i\neq\emptyset$ if and only if $0\notin\sum_{i\in I}\mathscr{C}_{K_i}$.*

Proof. Suppose that μ is a solution to (1). It is obvious from (1) that $0\notin\sum_{i\in I}\mathscr{C}_{K_i}$.

Conversely, suppose that $0\notin\sum_{i\in I}\mathscr{C}_{K_i}$. Since $\sum_{i\in I}\mathscr{C}_{K_i}$ is an open convex set, by the Separation Theorem, there exists a linear functional Λ on $C(\Omega)$ and a constant c such that for all $f\in\sum_{i\in I}\mathscr{C}_{K_i}$,

$$\Lambda(f)>c\geqslant 0.$$

(In fact, c must be zero.) Since $\sum_{i\in I}C_{K_i}$ contains all the positive continuous functions, Λ is a (strictly) positive linear functional. By the Reisz–Markov Representation Theorem, there is a unique regular Borel measure μ representing Λ. Thus, for all $f\in\sum_{i\in I}\mathscr{C}_{K_i}$, we have $\int_\Omega f\,d\mu>0$. Dividing by its own norm, we may let $\mu\in\mathscr{P}(\Omega)$. It follows from Eq. (1) that $\mu\in\bigcap_{i\in I}K_i$. □

We now summarize what we have obtained in the above discussion. For simplicity, we call an element $f\in\mathscr{C}_{K_i}$ a *positive bet* for agent i since it leads to positive expected gains under all his possible prior beliefs. First of all, Lemma 1 establishes a duality relationship between the prior beliefs (represented by K_i) and trading demands of bets (represented by $\mathscr{C}_{K_i}$, the set of all positive bets for agent i) for agent i. Secondly, we have obtained another duality relationship as follows. Let J be a subindex of I. It is clear that the set of all common priors among $i\in J$ corresponds to the *trading possibility set* $\sum_{i\in J}\mathscr{C}_{K_i}$. That is, $\bigcap$ and $\sum$ are dual operations. Furthermore, $\bigcap_{i\in J\subseteq I}K_i\neq\emptyset$ if and only if $0\notin\sum_{i\in J\subseteq I}\mathscr{C}_{K_i}$.

Now, an element $\mathbf{f}:I\to C(\Omega)$ summarizes all the individual bets. It is called a *trade* if for some finite $J\subseteq I$, $\sum_{i\in J}\mathbf{f}(i)=0$. The integral $\int_\Omega\mathbf{f}(i)\,\mathrm{d}\mu$ represents agent i's expected gain from the individual bet $\mathbf{f}(i)$ based on the prior μ. Theorem 1 says that a necessary and sufficient condition for the existence of a common prior is that, if it is always common knowledge that each agent makes positive bets, then it is impossible to find a trade among the agents.[5]

The following corollary is immediate from the definition of a trade:

Corollary 2. *There exists no common prior for all the agents if and only if there exists no common prior for some finite subset of agents.*

Notice that we have not introduced any information structure when we state the no trade principle as in Theorem 1. In a private information model, it is possible to identify three versions of Theorem 1 according to the timing of trading: ex ante, interim, or ex post. Technically speaking, both the ex ante and ex post cases can be

[5] We note that if there is no common prior, it might be impossible to find a trade in which every agent is involved (under the common knowledge assumption that agents only make positive bets). An easy example can be given when the number of agents is 3.

regarded as special cases of the interim case. Therefore, we shall only discuss the interim case.

We are now prepared to enrich the model by adding an interim stage, in which each agent obtains private information on the state of the world. Depending on his private information, he forms posterior beliefs over Ω according to Bayes' rule, i.e., they are all Bayesian. From the interim viewpoint, the private information and posterior beliefs of each agent are all given.

Formally, we define a *type space* to be a tuple $\{(\Omega, (\mathscr{F}_i, t_i)_{i\in I})\}$ where for each $i\in I$, $\mathscr{F}_i$ is a sub-σ-algebra of $\mathscr{F}$ and $t_i : \Omega \times \mathscr{F} \to [0,1]$ is a *type* (that consists of *posterior beliefs*) for agent i, that is,

(a) for each $\omega\in\Omega$, $t_i(\omega,\cdot)\in\mathscr{P}(\Omega)$;
(b) for each $E\in\mathscr{F}$, $t_i(\cdot,E)$ is a $\mathscr{F}_i$-measurable function;
(c) (properness) for each $B\in\mathscr{F}_i$, $t_i(\omega,B) = \begin{cases} 1 & \text{if } \omega\in B, \\ 0 & \text{if } \omega\notin B. \end{cases}$

What is the set of all possible priors for agent i from the interim viewpoint? Following Harsanyi [6], any prior of an agent should be consistent with his type according to Bayes' rule. Thus, in the interim case, a *prior* for agent $i\in I$ becomes a measure $\mu\in\mathscr{P}(\Omega)$ such that for each $E\in\mathscr{F}$, $B\in\mathscr{F}_i$,

$$\mu(E\cap B) = \int_B t_i(\omega, E)\, d\mu(\omega).$$

Let P_i be the set of all priors of agent i. It is clear that P_i is a convex set in $\mathscr{P}(\Omega)$. Notice that P_i may not be compact.

Definition 1. A *common prior* on the type space $\{(\Omega, (\mathscr{F}_i, t_i)_{i\in I})\}$ is an element of $\bigcap_{i\in I} P_i$.

The following lemma states a necessary and sufficient condition for an element to be in P_i. It will be very useful. In fact, Milgrom and Stokey [7] have used it as a key step in proving the no trade theorem.

Lemma 2. *For each* $i\in I$, $\mu\in P_i$ *if and only if for any bounded Borel measurable function* f, *we have*

$$\int_\Omega \int_\Omega f(\bar{\omega}) t_i(\omega, d\bar{\omega})\, d\mu(\omega) = \int_\Omega f\, d\mu. \tag{2}$$

For any set A in a topological vector space, we let *con* A be the convex hull generated by elements in A. Hence, *con* A consists of all finite convex combinations of elements from A. The closed convex hull of A, denoted by $\overline{con}$ A, is the closure of *con* A. We have the following observations:

Proposition 1. *con* $\{t_i(\omega,\cdot)\colon \omega\in\Omega\} \subseteq P_i \subseteq \overline{con}\ \{t_i(\omega,\cdot)\colon \omega\in\Omega\}$.

An important implication of Proposition 1 is that the extreme points of the set of all possible priors for a given agent can be exactly described by his type.

Proposition 2. *For each $i \in I$, we have*

$$\mathscr{C}_{P_i} = \left\{ f \in C(\Omega) \colon \int_\Omega f(\bar{\omega}) t_i(\omega, d\bar{\omega}) > 0 \quad \text{for all } \omega \in \Omega \right\}.$$

Proposition 2 shows a tight connection between the posterior beliefs of an agent and all his possible prior beliefs. It shows that both induce the same trading demand set. In other words, Bayes' rule is *dynamically consistent*.

Obviously, if P_i is finitely generated from $\{t_i(\omega, \cdot) \colon \omega \in \Omega\}$, then *con* $\{t_i(\omega, \cdot) \colon \omega \in \Omega\} = \overline{con}$ $\{t_i(\omega, \cdot) \colon \omega \in \Omega\}$. Hence, we have the following corollary:

Corollary 3. *If P_i is finitely generated from $\{t_i(\omega, \cdot) \colon \omega \in \Omega\}$, then P_i is equal to $\overline{con}\{t_i(\omega, \cdot) \colon \omega \in \Omega\}$.*

Credit should be given to Samet [12], where it is noted that Corollary 3 is obvious when Ω is finite. However, the following example shows that, in general, the inclusions in Proposition 1 are strict.

Example 1. Let $\Omega = [0, 1]$ and μ be the point mass measure at 1. Let μ_0 be any distribution with support on $A_0 = \{0, 1\}$ such that $\mu_0(\{0\}) > 0$. For each natural number k, let us define μ_k as any distribution with support on $A_k = (1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}]$.

We claim that $\mu_k \to \mu$ as $k \to \infty$. To prove it, let us take an $f \in C(\Omega)$ and an $\varepsilon > 0$. By continuity, there exists a K such that, for all $k \geqslant K$ and for all $x \in A_k$, we have $|f(x) - f(1)| < \varepsilon$. It follows that for each $k \geqslant K$,

$$f(1) - \varepsilon < \int_\Omega f \, d\mu_k < f(1) + \varepsilon.$$

Thus, $\int_\Omega f \, d\mu_k \to f(1) = \int_\Omega f \, d\mu$.

Consider the sub-σ-algebra $\mathscr{F}_i$ generated by the partition $\{A_0, A_1, A_2, \ldots, A_k, \ldots\}$. Suppose that agent i has the following type function with respect to $\mathscr{F}_i$:

$$t_i(\omega, \cdot) = \mu_k \text{ if } \omega \in A_k, \quad k \geqslant 0.$$

Then, $\mu \in \overline{con}$ $\{t_i(\omega, \cdot) \colon \omega \in \Omega\}$. Now, let $E = \{1\}$. We have

$$\int_{A_0} t_i(\omega, E) \, d\mu(\omega) = \mu_0(E) < 1 = \mu(E \cap A_0).$$

This contradicts Bayes' rule. Thus, $\mu \notin P_i$.

It is obvious that the measure $\sum_{i=1}^{\infty} \frac{1}{2^k} \mu_k$ is an element of P_i but not an element of *con* $\{t_i(\omega, \cdot) \colon \omega \in \Omega\}$. Consequently, both inclusions in Proposition 1 are strict. $\square$

In order to apply the no trade principle, we make the following technical assumption. It is crucial in our analysis since it implies that $P_i = \overline{con}\ \{t_i(\omega, \cdot): \omega \in \Omega\}$.

Assumption 1. For each $i \in I$, P_i is a closed (hence compact) set in $\mathscr{P}(\Omega)$.

It is clear that Bayes' rule continues to hold as we pass to strong limits. That is, if for each n, μ_n satisfies Bayes' rule and $\mu_n(E) \to \mu(E)$ for all $E \in \mathscr{F}$, then μ satisfies Bayes' rule. Assumption 1 is essentially saying that we still want to keep Bayes' rule as we pass to weak* limits.

The following result follows immediately from Theorem 1 and Proposition 2.

Theorem 2 (Interim no trade principle). *Supposing that Assumption* 1 *holds, a necessary and sufficient condition for the type space* $\{(\Omega, (\mathscr{F}_i, t_i)_{i \in I})\}$ *to have a common prior is: if it is always common knowledge that each agent expects positive gains with respect to his posterior beliefs at the interim stage, then there is no trade.*

The above theorem should be compared to Lemmas A2(iv) and A3 in [8] (see also the remarks after Theorem 3.1 there). We note that Morris [8] is actually defining ex ante consistency conditions on given prior beliefs that lead to the same posterior beliefs, instead of defining interim consistency conditions.

3. Concluding remarks

Remark 2. It is important to notice that in deriving the interim no trade principle from the no trade principle, all we need is to ensure that Bayes' rule is dynamically consistent (Proposition 2). This suggests that the interim no trade principle should still be valid in a more general private information model, in which agents may not be Bayesians and information may not be partitional.

Remark 3. This paper does not allow agents' preferences to deviate from the usual assumptions of expected utility theory. It would be interesting to study when the no trade principle continues to hold without expected utility. In fact, there are already some papers which study topics closely related to this question. For example, Halevy [5] provides sufficient and necessary conditions on preferences (notice that his conditions are not expressed in terms of beliefs) for the impossibility of speculative trade when agents are dynamically consistent but their preferences do not satisfy some expected utility assumptions.

Remark 4. We have made no restriction on the number of agents. The index set $I \neq \emptyset$ can be anything, finite or infinite. In particular, no algebraic or topological structure has been imposed on I. This is crucial in some economic problems. For example, it is well-known that, while the core equivalence theorem holds for the continuum case, it depends heavily on the well-ordered structure of the unit interval.

Remark 5. For the sake of the Separation Theorem, we have to assume that the set of all possible priors of each agent is compact. However, what can we do if we want to relax the compactness assumption made with regard to Ω?

Firstly, the most natural extension to consider is actually $C_c(\Omega)$, the space of all continuous functions on a locally compact Hausdorff space Ω with compact support. It is well-known that its norm dual is $rca(\mathscr{F})$, the space of all regular signed Borel measures of bounded variation. Certainly, $rca(\mathscr{F})$ contains $\mathscr{P}(\Omega)$ as a subset. It is easy to show that Theorem 1 remains valid. We are not doing it simply because of economic reasons. It seems artificial to restrict trading demands as continuous functions with compact support.

Secondly, if we only want to consider bounded continuous functions on a normal space, then we can look at its dual space which consists of all finitely additive, normal signed Borel measures. We suggest that it may be possible to establish a no trade principle for some non-compact type spaces.

The aforementioned research is important because an affirmative answer to it would give additional insight into the role played by compactness assumptions. It would mean that individual duality and individual compactness are the two crucial elements for establishing the no trade principle.[6]

Appendix A. Generalization

In the analysis of Section 2, we have implicitly assumed that trade is rejected if an agent is indifferent between accepting and rejecting. This assumption is not necessary because a similar individual duality relationship still holds if we replace positive bets by *non-negative* bets. In other words, for each $i \in I$, we can also write

$$K_i = \left\{ \mu \in \mathscr{P}(\Omega) \colon \int_\Omega f \, d\mu \geqslant 0 \text{ for all } f \in \overline{\mathscr{C}_{K_i}} \right\}, \tag{A.1}$$

where $\overline{\mathscr{C}_{K_i}} = \{ f \in C(\Omega) \colon \int_\Omega f \, d\mu \geqslant 0 \text{ for all } \mu \in K_i \}$.

The proof of (A.1) will be omitted since the arguments are similar to those of Lemma 1.

Definition A.1. An element $\mathbf{f} : I \to C(\Omega)$ is a (respectively, *strictly*) *Q-Pareto improving trade* if there exists a finite $J \subseteq I$ such that

(a) $\sum_{i \in J} \mathbf{f}(i) = 0$,
(b) $\mathbf{f}(i) \in \overline{\mathscr{C}_{K_i}}$ for all $i \in J$ and $\mathbf{f}(j) \in \mathscr{C}_{K_j}$ for some $j \in J$. (respectively, $\mathbf{f}(i) \in \mathscr{C}_{K_i}$ for all $i \in J$.)

[6] Interestingly, Feinberg [4, p. 152] is able to give an example in which there is no common prior and yet there is no trade involving bounded bets. However, it seems that the sets of prior beliefs for both agents in his example are not compact.

Note that in a strictly Q-Pareto improving trade, we do not require that each agent $i \in I$ has positive expected gains. As a result, we have made no attempt to generalize Lemmas A3, A4 and A5 in Morris [8].

It is easy to see that Theorem 1, together with Eq. (A.1), leads to the following alternative version of the no trade principle:

Theorem 1A. *A common prior exists if and only if no* (*strictly*) *Q-Pareto improving trade exists.*

Proof. Let μ be a common prior. Suppose that there is a Q-Pareto improving trade $\mathbf{f} : I \to C(\Omega)$. Then, there is a finite $J \subseteq I$ such that conditions (a) and (b) in Definition 1 are satisfied. Condition (a) implies that $\sum_{i \in J} \int_\Omega \mathbf{f}(i)\, d\mu = 0$. However, by Lemma 1(ii) and Eq. (A.1), condition (b) implies that $\sum_{i \in J} \int_\Omega \mathbf{f}(i)\, d\mu > 0$, a contradiction. Thus, there exists no Q-Pareto improving trade, hence no strictly Q-Pareto improving trade. The claim in the theorem follows from Theorem 1 immediately. □

We can also relax the risk-neutrality assumption and extend to an exchange economy environment with L commodities, where L is a fixed natural number. Moreover, we may allow a countable number of payoff-relevant states. Most ideas of the proofs in [8] can be generalized. We shall, therefore, state the most general theorem but omit the detailed proofs. The required proofs can still be found in [10]. Without loss of generality, we assume that the set of payoff-relevant states is a singleton in the following presentation.

Let us introduce some notations. The standard notation $C(\Omega, R^L)$ represents the class of all continuous functions from Ω into R^L. We shall restrict the utility function of each agent to be an element of $\mathscr{U}$ that consists of all strictly increasing, quasi-concave and continuous functions on R^L. For each $i \in I$, we let $e_i \in C(\Omega, R^L)$ be the endowment of agent i. Let us denote the endowment function by $e : I \to C(\Omega, R^L)$, i.e., $e(i) = e_i$ for $i \in I$.

Let $u_i \in \mathscr{U}$ be a given utility function of agent i. For each $i \in I$, consider the function $\Delta u_i : C(\Omega, R^L) \to C(\Omega)$ defined as follows. For each $f \in C(\Omega, R^L)$, and $\omega \in \Omega$, let

$$\Delta u_i(f)(\omega) = u_i(e_i(\omega) + f(\omega)) - u_i(e_i(\omega)).$$

Given $Q = \{K_i\}_{i \in I}$, we can re-define agent i's *set of trading demands* as

$$\mathscr{D}_{K_i} = \left\{ f \in C(\Omega, R^L) \colon \int_\Omega \Delta u_i(f)\, d\mu > 0 \text{ for all } \mu \in K_i \right\}$$

or

$$\overline{\mathscr{D}_{K_i}} = \left\{ f \in C(\Omega, R^L) \colon \int_\Omega \Delta u_i(f)\, d\mu \geqslant 0 \text{ for all } \mu \in K_i \right\}$$

according to the corresponding common knowledge assumption regarding trading behavior. We can also define Q-Pareto improving trades accordingly. We say that e is (strictly) Q-Pareto *efficient* if there exists no (strictly) Q-Pareto improving trade.

A *constant* trade is a trade that is independent of the state of nature. The endowment function e is *initially efficient* if there is no Q-Pareto improving constant trade. We have the following theorem:

Theorem A.1. *Suppose that e is initially efficient, and that for each $i \in I$, $u_i \in \mathscr{U}$ is differentiable at the constant endowment $e_i \in R^L$. Then, $0 \notin \sum_{i \in I} \mathscr{D}_{K_i}$ implies that $\bigcap_{i \in I} K_i \neq \emptyset$. Suppose, in addition, that each u_i is concave. Then, the existence of a common prior implies that e is Q-Pareto efficient.*

Corollary A.1. *Suppose that e is initially efficient, and that for each $i \in I$, $u_i \in \mathscr{U}$ is concave, and differentiable at the constant endowment $e_i \in R^L$. Then, a common prior exists if and only if e is (strictly) Q-Pareto efficient.*

Appendix B. Proofs

Proof of Lemma 1. (i) Let $f \in \mathscr{C}_K$. Since K is compact, we may let $2\varepsilon = \inf_{\bar{\mu} \in K} \int_\Omega f\, d\bar{\mu} = \min_{\bar{\mu} \in K} \int_\Omega f\, d\bar{\mu} > 0$. For any $\mu \in K$ and for any $g \in C(\Omega)$ such that $||g - f|| < \varepsilon$, we have

$$\int_\Omega g\, d\mu \geqslant \int_\Omega f\, d\mu - \int_\Omega |g - f|\, d\mu \geqslant 2\varepsilon - \varepsilon > 0.$$

Then $g \in \mathscr{C}_K$ and $\mathscr{C}_K$ is an open set.

(ii) By the Separation Theorem, $\mu \in K$ if and only if for all $f \in C(\Omega)$ and constants c such that $\inf_{\bar{\mu} \in K} \int_\Omega f\, d\bar{\mu} > c$, we have $\int_\Omega f\, d\mu \geqslant c$. By taking c arbitrarily close to $\inf_{\bar{\mu} \in K} \int_\Omega f\, d\bar{\mu}$, we have $\int_\Omega f\, d\mu \geqslant \inf_{\bar{\mu} \in K} \int_\Omega f\, d\bar{\mu} > c$. Now by replacing f with $f - c$, we may assume c to be zero. It follows that $\mu \in K$ if and only if for all $f \in \mathscr{C}_K$, $\int_\Omega f\, d\mu > 0$. This completes the proof of the lemma. □

Proof of Remark 1. We shall prove the results by assuming that $\mathscr{P}$ is weak* metrizable. Replacing sequences by nets can drop the assumption.

It follows directly from the Separation Theorem (and the Reisz–Markov Representation Theorem) that K_C is non-empty. That K_C is a convex set is obvious. To show that K_C is a compact set, we have to prove that it is closed. Let μ_n be a sequence in K_C converging to μ_0. Then for every $f \in C(\Omega)$, $\lim_{n \to \infty} \int_\Omega f\, d\mu_n = \int_\Omega f\, d\mu_0$. Take an $f \in C$. Since C is open, there exists an $\varepsilon > 0$ such that $f - \varepsilon \in C$. It follows that for each n, $\int_\Omega (f - \varepsilon)\, d\mu_n > 0$. Hence, $\int_\Omega f\, d\mu_0 \geqslant \varepsilon > 0$ and so $\mu_0 \in K_C$. This proves that K_C is a compact subset of $\mathscr{P}(\Omega)$. Now, it is clear from the definition of K_C that

$$C \subseteq \left\{ f \in C(\Omega) : \int_\Omega f\, d\mu > 0 \text{ for all } \mu \in K_C \right\} \equiv D.$$

Suppose that $f \notin C$. By the Separation Theorem and the fact that C is an open convex cone not containing 0, there is an $\bar{\mu} \in \mathscr{P}(\Omega)$ (as in the proof of Theorem 1, we need the

Reisz–Markov Representation Theorem here) such that for each $g \in C$, $\int_\Omega g\, d\bar{\mu} > 0 \geqslant \int_\Omega f\, d\bar{\mu}$. The first inequality shows that $\bar{\mu} \in K_C$. Then it follows from the second inequality that $f \notin D$. Hence, $C = D$. □

Proof of Lemma 2. Let $\mu \in P_i$. It suffices to prove Eq. (2) for all simple functions. Let $f = \sum_{i=1}^{k} \alpha_i \chi_{E_i}$ where the E_i are mutually disjoint measurable sets in $\mathscr{F}$, $\alpha_i \in R$ and χ_E stands for the characteristic function of the set E. Then

$$\int_\Omega \int_\Omega f(\bar{\omega}) t_i(\omega, d\bar{\omega}) d\mu(\omega) = \int_\Omega \left[\sum_{i=1}^{k} \alpha_i \int_{E_i} t_i(\omega, d\bar{\omega}) \right] d\mu(\omega).$$

$$= \sum_{i=1}^{k} \alpha_i \int_\Omega t_i(\omega, E_i) d\mu(\omega) = \sum_{i=1}^{k} \alpha_i \mu(E_i) = \int_\Omega f\, d\mu.$$

Conversely, for each $E \in \mathscr{F}$ and $B \in \mathscr{F}_i$, putting $f = \chi_{E \cap B}$ in Eq. (2) reduces to Bayes' rule. □

Proof of Proposition 1. We first prove that for each $\omega_o \in \Omega$, $t_i(\omega_o, \cdot) \in P_i$. Let $E \in \mathscr{F}$ and $B \in \mathscr{F}_i$. Note that if $\omega \in B$, then

$$t_i(\omega, E) = t_i(\omega, E \cap B) + t_i(\omega, E \cap B') = t_i(\omega, E \cap B).$$

Now fix $\omega_o \in \Omega$. Without loss of generality, we may assume that $\omega_o \in B$. Let us then define the following two $\mathscr{F}_i$-measurable sets:

$$B_1 = \{\omega \in B\colon\ t_i(\omega, E) = t_i(\omega_o, E)\} \quad \text{and} \quad B_2 = \{\omega \in B\colon\ t_i(\omega, E) \neq t_i(\omega_o, E)\}.$$

Then $t_i(\omega_o, B_1) = 1$ and $t_i(\omega_o, B_2) = 0$. We have

$$\int_B t_i(\omega, E) t_i(\omega_o, d\omega) = \int_{B_1} t_i(\omega_o, E) t_i(\omega_o, d\omega) = \int_B t_i(\omega_o, E) t_i(\omega_o, d\omega)$$

$$= t_i(\omega_o, E \cap B).$$

It follows that $t_i(\omega_o, \cdot)$ is a prior for agent i. Since P_i is a convex set, we must have *con* $\{t_i(\omega, \cdot)\colon\ \omega \in \Omega\} \subseteq P_i$.

Now, suppose that $\mu_0 \notin \overline{con}\ \{t_i(\omega, \cdot)\colon\ \omega \in \Omega\}$. By the Separation Theorem, there exists a function $f \in C(\Omega)$ and a constant c such that $\int_\Omega f\, d\mu_0 > c$ and

$$\int_\Omega f(\bar{\omega}) t_i(\omega, d\bar{\omega}) < c \quad \text{for each } \omega \in \Omega. \tag{B.1}$$

Eq. (B.1) implies that

$$\int_\Omega \int_\Omega f(\bar{\omega}) t_i(\omega, d\bar{\omega}) d\mu_0(\omega) < c.$$

It follows from Lemma 2 that $\mu_0 \notin P_i$. This completes the proof of the proposition. □

Proof of Proposition 2. Let us denote

$$D = \left\{ f \in C(\Omega): \int_\Omega f(\bar{\omega}) t_i(\omega, d\bar{\omega}) > 0 \text{ for all } \omega \in \Omega \right\}.$$

From Proposition 1, it is obvious that $\mathscr{C}_{P_i} \subseteq D$. Supposing that $f \in D$, then for each $\mu \in P_i$, Lemma 2 implies that

$$\int_\Omega f \, d\mu = \int_\Omega \int_\Omega f(\bar{\omega}) t_i(\omega, \bar{\omega}) \, d\mu(\omega) > 0.$$

Thus, $\mathscr{C}_{P_i} = D$. □

References

[1] A. Billot, A. Chateauneue, I. Gilboa, J.-M. Tallon, Sharing beliefs: between agreeing and disagreeing, Econometrica 68 (2000) 685–694.

[2] V.G. Boltyanski, On separation systems of convex cones, Izv. AN Arm. SSR (Matem.) 7 (1972) 325–333 (in Russian).

[3] Y. Feinberg, A converse to the agreement theorem, Discussion Paper 83, Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, 1995.

[4] Y. Feinberg, Characterizing common priors in the form of posteriors, J. Econom. Theory 91 (2) (2000) 127–179.

[5] Y. Halevy, The possibility of speculative trade between dynamically consistent agents, Working paper, The University of British Columbia, 2000.

[6] J. Harsanyi, Games with incomplete information played by 'Bayesian' players III, Management Sci. 14 (1968) 486–502.

[7] P. Milgrom, N. Stokey, Information, trade and common knowledge, J. Econom. Theory 26 (1982) 17–27.

[8] S. Morris, Trade with heterogeneous prior beliefs and asymmetric information, Econometrica 62 (1994) 1327–1347.

[9] S. Morris, The common prior assumption in economic theory, Econom. Philos. 11 (1995) 227–253.

[10] M.-C. Ng, On the duality of prior beliefs and trading demands, Working paper, Institute of Economics, Academia Sinica, Taiwan, 1999.

[11] M. Rubinstein, Securities market efficiency in an Arrow-Debreu economy, Amer. Econom. Rev. 65 (1975) 812–824.

[12] D. Samet, Common priors and separation of convex sets, Games Econom. Behav. 24 (1998) 172–174.

[13] J. Sebenius, J. Geanakoplos, Don't bet on it: contingent agreements with asymmetric information, J. Amer. Statist. Assoc. 78 (1983) 424–426.

[14] M. Sion, On general minimax theorems, Pacific J. Math. 8 (1958) 171–176.

[15] J. Tirole, On the possibility of speculation under rational expectations, Econometrica 50 (1982) 1163–1181.