

Secured lower bound, composition up, and minimal rights first for bankruptcy problems

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Abstract

We study the implications of secured lower bound when imposed together with minimal rights first or composition up for the resolution of conflicting claims. We show that the Talmud rule is the only rule satisfying secured lower bound, minimal rights first, and consistency. In addition, we show that if minimal rights first and consistency in the above characterization were replaced with composition up and null claims consistency, respectively, the constrained equal awards rule stands out as the only acceptable rule. *Journal of Economic Literature* Classification Number: C79; D63; D74.

Keywords: Claims problems; the constrained equal awards rule; the Talmud rule; composition property; consistency

1 Introduction

We consider the problem of distributing the liquidation value of a bankrupt firm among its creditors. How should the value be distributed? A “rule” is a function that associates with each such problem a division of the liquidation

value, which we call an “awards vector.” A number of desirable properties of rules have been proposed for this problem, motivated by fairness, participation, or incentive considerations. The literature devoted to the search for rules satisfying these properties, singly and in various combinations, was initiated by O’Neill (1982).¹

What should the minimal award to a creditor be? The most natural requirement is of course that a creditor should receive a non-negative amount. However, this requirement is very weak and here we consider a recent suggestion, “secured lower bound,” proposed by Moreno-Ternero and Villar (2004).² The requirement is that each creditor should receive at least the amount obtained by first truncating his claim at the liquidation value, and then dividing this truncated claim by the number of creditors.³ The idea of truncating claims in this manner plays an important role in the literature and a number of axiomatic characterizations of well-known rules involve a requirement of invariance with respect to such truncation (e.g., Dagan, 1996; Hokari and Thomson, 2003).

Our purpose here is to explore the implications of *secured lower bound* when imposed together with each of the following two composition properties. For each problem, the awards vector can be calculated either directly, or in two steps, as follows: we first attribute to each creditor “his minimal right” (the maximum of zero and the difference between the sum of the claims of all other creditors and the liquidation value); then after revising their claims down by their minimal rights, we divide the remainder according to the revised claims vector. The first composition property, *minimal rights first* (e.g., Curiel, Maschler, and Tijs, 1987), says that both ways of proceeding should yield the same awards vector.⁴ The second composition property has to do with possible

¹For a comprehensive survey of this literature, see Thomson (2003).

²Moreno-Ternero and Villar (2004) refer to it as *securement*.

³There are several ways to define a pre-specified proportion of the truncated claim of a creditor. We here adopt the inverse of the number of creditors since it is the highest that preserves *compatibility*: the sum of all awards should be at most as large as the liquidation value. For a detailed discussion on this idea, see Dominguez and Thomson (2006).

⁴Curiel, Maschler, and Tijs (1987) refer to it as the *minimal rights property*. Dominguez and Thomson (2006) formulate a similar composition property concerning the lower bound on creditors’ awards in the definition of *secured lower bound*. Dominguez (2006) proposes another lower bound on creditors’ awards. Based on this minimal award to a creditor, he formulates another composition property.

changes in the liquidation value. Suppose that after an awards vector is chosen for some problem, the liquidation value is found to be greater than initially thought. There are two ways to handle this increase. One is to cancel the initial division and recalculate the awards for the revised liquidation value. The other is to let creditors keep their initial awards, revise their claims down by these awards, and divide the incremental amount according to the revised claims. *Composition up* (e.g., Young, 1988) says that both ways of proceeding should yield the same awards vector.⁵

Are there rules satisfying *secured lower bound* as well as *minimal rights first* or *composition up*? It is known that the Talmud rule (e.g., Aumann and Maschler, 1985), proposed to rationalize the recommendations made in the Talmud for several numerical examples, satisfies *secured lower bound* and *minimal rights first*. However, it is not the only rule satisfying the two properties. Others are the “random arrival” (e.g., O’Neill, 1982) and “adjusted proportional” rules (e.g., Curiel, Maschler, and Tijs, 1987). Note that in the two-creditor case, these rules coincide with “concede-and-divide” (e.g., Aumann and Maschler, 1985).⁶ Based on the above observation, one may ask whether a rule satisfying *secured lower bound* and *minimal rights first* must coincide with concede-and-divide in the two-creditor case. We show that this is indeed the case (Proposition 1). Note that concede-and-divide violates *composition up*, whereas the constrained equal awards rule (e.g., Aumann and Maschler, 1985), which assigns equal amounts to all creditors subject to no one receiving more than his claim, satisfies the property. Moreover, it can be shown that the constrained equal awards (CEA) rule satisfies *secured lower bound*. Is there any rule other than the CEA rule satisfying *secured lower bound* and *composition up*? Surprisingly, in the two-creditor case, the answer is no (Proposition 2).

For more than two creditors, we lose the uniqueness parts of Propositions 1 and 2. We next generalize the two results to more than two creditors in a “consistent” manner. The property on which these generalizations are based

⁵Young (1988) refers to it as *composition*.

⁶This rule is the two-creditor rule defined by first attributing to each creditor his minimal right, called the conceded part, and then dividing the remainder, called the contested part, equally (after being adjusted down by the minimal rights, and then truncated at the remainder, claims are equal. Thus, equal division is very natural). The terminology is borrowed from Thomson (2001a).

is *consistency* (e.g., Aumann and Maschler, 1985). Consider a bankruptcy problem and an awards vector chosen by a rule for it. Now, imagine that some creditors leave with their awards, and reassess the situation from the viewpoint of the remaining creditors. The requirement is that in the reduced problem faced by the remaining creditors, the rule should attribute to each of them the same amount as initially.⁷

It is known that the Talmud rule is *consistent*. Moreover, as we show, the rule is the only rule satisfying *secured lower bound*, *minimal rights first*, and *consistency* (Theorem 1). Note that the CEA rule is *consistent*. Is there any *consistent* rule other than the CEA rule satisfying *secured lower bound* and *composition up*. The answer is negative. Moreover, we show that this characterization of the CEA rule can be strengthened by weakening *consistency* to *null claims consistency* (e.g., Thomson, 2003): the departure of creditors whose claims are zero should not affect the awards to the remaining creditors. Namely, the CEA rule is the only rule satisfying *secured lower bound*, *composition up*, and *null claims consistency* (Theorem 2).

The rest of the paper is organized as follows. Section 2 introduces the model, the central properties, and the important rules. Section 3 presents our results. Section 4 shows independence of the properties listed in each of our results, derives the “dual” characterizations of our results, compares our results with existing ones, and makes a final remark.

2 The model, the central properties, and the important rules

There is an infinite set of “potential” creditors, indexed by the natural numbers \mathbb{N} . Let \mathcal{N} be the class of non-empty and finite subsets of \mathbb{N} . Given $N \in \mathcal{N}$ and $i \in N$, let c_i be **creditor i 's claim** and $c \equiv (c_i)_{i \in N}$ the claims vector. The **liquidation value** E of a bankrupt firm has to be divided among the creditors N . A bankruptcy problem for N , or simply a **problem for N** , is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$.⁸ Let \mathcal{C}^N be the class of all problems for N . An **awards vector** for $(c, E) \in \mathcal{C}^N$ is a vector $x \in \mathbb{R}^N$ such that

⁷For a survey of the literature on *consistency* and its converse, see Thomson (2000).

⁸By \mathbb{R}_+^N we denote the Cartesian product of $|N|$ copies of \mathbb{R}_+ , indexed by the elements of N . Vector inequalities: $x \geq y$, $x \geq y$, and $x > y$.

$0 \leq x \leq c$ and $\sum_{i \in N} x_i = E$. Let $X(c, E)$ be the set of awards vectors of (c, E) . A **rule** is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a vector in $X(c, E)$. Our generic notation for rules is φ . For each group $N' \subset N$, we write $c_{N'}$ for $(c_i)_{i \in N'}$, $\varphi_{N'}(c, E)$ for $(\varphi_i(c, E))_{i \in N'}$, and so on. Also, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, the “minimal right of creditor i ” is the quantity $m_i(c, E) \equiv \max \left\{ E - \sum_{j \in N \setminus \{i\}} c_j, 0 \right\}$, and $m(c, E) \equiv (m_i(c, E))_{i \in N}$ is the minimal rights vector.

We now introduce the rules that are central to our analysis.

Constrained equal awards rule, CEA: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$CEA_i(c, E) \equiv \min \{c_i, \lambda\},$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.

Talmud rule, T: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$T_i(c, E) \equiv \begin{cases} \min \left\{ \frac{c_i}{2}, \lambda \right\} & \text{if } \sum_{i \in N} \frac{c_i}{2} \geq E; \\ c_i - \min \left\{ \frac{c_i}{2}, \lambda \right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} T_i(c, E) = E$.

We study the following properties informally defined in the introduction.

Secured lower bound: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $\varphi_i(c, E) \geq \frac{1}{|N|} \min \{c_i, E\}$.

Minimal rights first: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, we have $\varphi(c, E) = m(c, E) + \varphi(c - m(c, E), E - \sum_{i \in N} m_i(c, E))$.

Composition up: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, we have $\varphi(c, E') = \varphi(c, E) + \varphi(c - \varphi(c, E), E' - E)$.

Consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $x \equiv \varphi(c, E)$, then $x_{N'} = \varphi(c_{N'}, \sum_{i \in N'} x_i)$.

Null claims consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $c_{N \setminus N'} = 0$, then $\varphi_{N'}(c, E) = \varphi(c_{N'}, E)$.

Remark: Clearly, *consistency* implies *null claims consistency*. Also, *composition up* implies “resource monotonicity,” and *resource monotonicity* implies “resource continuity.”⁹ Thus, *composition up* implies *resource continuity*. However, *minimal rights first* does not imply *resource monotonicity*. For a proof of this fact, see Thomson (2001a). Besides, Chun (1999) shows that *consistency* and *resource monotonicity* together imply “converse consistency”¹⁰.

3 Results

We explore the implications of *secured lower bound* when imposed together with different composition properties. We start with *minimal rights first*. A number of rules satisfy *secured lower bound* and *minimal rights first*. Examples are the Talmud, “adjusted proportional,” and “random arrival” rules. In the two-creditor case, these rules coincide with the following rule informally defined in the introduction.

Concede-and-Divide, CD: For $|N| = 2$. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$CD_i(c, E) \equiv \max \{E - c_{N \setminus \{i\}}, 0\} + \frac{E - \sum_{k \in N} \max \{E - c_k, 0\}}{2}.$$

Based on the above observation, it is natural to ask whether a rule satisfying *secured lower bound* and *minimal rights first* must coincide with CD in the two-creditor case. We show that this is indeed the case. Moreno-Tenero and Villar (2006) characterize CD by imposing *minimal rights first* and a weaker version of *secured lower bound*. Since our proof is essentially the same as theirs, we omit our proof.¹¹

⁹*Resource monotonicity* says that if the liquidation value increases, each creditor’s award should be at least as large as it was initially. *Resource continuity* says that small changes in the liquidation value should not lead to large changes in the awards vector.

¹⁰This property is a converse of *consistency* and says that whenever an awards vector for some problem is such that its restriction to each two-creditor group is the choice a rule makes for the associated reduced problem, then it is the choice the rule makes for the initial problem. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $x \in X(c, E)$, if for each $N' \subset N$ with $|N'| = 2$, $x_{N'} \equiv \varphi(c_{N'}, \sum_{i \in N'} x_i)$, then $x = \varphi(c, E)$.

¹¹For more characterizations of CD, readers are referred to Dagan (1996) and Moreno-Tenero (2006).

Proposition 1 For $|N| = 2$. The rule CD is the only rule satisfying *secured lower bound* and *minimal rights first*.

For more than two creditors, we lose the uniqueness part of Proposition 1. Note that the Talmud rule is *consistent*. Besides, it satisfies *secured lower bound* and *minimal rights first*. Are there such rules other than the Talmud rule? We show that the answer is no. Namely, the Talmud rule is the only rule satisfying *secured lower bound*, *minimal rights first*, and *consistency*. Since this characterization of the Talmud rule is a consequence of Corollary 1 (v) in Moreno-Ternero and Villar (2006), we omit the proof.¹²

Theorem 1 The Talmud rule is the only rule satisfying *secured lower bound*, *minimal rights first*, and *consistency*.

We next study the implications of *secured lower bound* when imposed in conjunction with *composition up*. It is known that CD violates *composition up*. Thus, the Talmud, random arrival, and adjusted proportional rules also violate the property. Note that the CEA rule satisfies *composition up*. Moreover, it can be shown that the CEA rule satisfies *secured lower bound*. One may wonder whether there are such rules other than the CEA rule satisfying *secured lower bound* and *composition up*. We show that in the two-creditor case, the answer is no.

Proposition 2 For $|N| = 2$. The CEA rule is the only rule satisfying *secured lower bound* and *composition up*.

Proof. Clearly, the CEA rule satisfies the two properties. Conversely, let φ be a two-creditor rule satisfying the properties. Without loss of generality, let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2$. Let $y \equiv \varphi(c, E)$ and $x \equiv CEA(c, E)$. To show that $y = x$, we consider three cases.

Case 1: $E \leq c_1$. Since $c_1 \leq c_2$, by *secured lower bound*, $y \geq (\frac{E}{2}, \frac{E}{2}) = x$. Since $\sum_{i \in N} y_i = E$, it follows that $y = x$.

¹²Alternatively, our characterization of the Talmud rule can be obtained by exploiting Proposition 1 and the Elevator Lemma (e.g., Thomson, 2000): If a rule φ is *consistent* and coincides with a *conversely consistent* rule φ' in the two-creditor case, then φ coincides with φ' in general.

Case 2: $c_1 < E \leq 2c_1$. Let $c_1^0 \equiv c_1$, $c_1^1 \equiv \frac{c_1^0}{2}$, $c_1^2 \equiv \frac{c_1^1}{2}$, and so on. Let $k \in \mathbb{N}$ be such that $k \geq 1$ and $\sum_{p=0}^{k-1} c_1^p < E \leq \sum_{p=0}^k c_1^p$. The proof is by induction on k .

Subcase 2.1: $k = 1$. Then, $c_1^0 < E \leq c_1^0 + c_1^1$. Let $E^* \equiv c_1^0$. By *composition up*, $y = \varphi(c, E^*) + \varphi(c - \varphi(c, E^*), E - E^*)$. By Case 1, $\varphi(c, E^*) = (\frac{E^*}{2}, \frac{E^*}{2})$. Since $E \leq c_1^0 + c_1^1$, it follows that $E - E^* \leq c_1 - \varphi_1(c, E^*)$. Note that $c_1 - \varphi_1(c, E^*) \leq c_2 - \varphi_2(c, E^*)$. By Case 1, $\varphi(c - \varphi(c, E^*), E - E^*) = (\frac{E - E^*}{2}, \frac{E - E^*}{2})$. Thus, $y = (\frac{E}{2}, \frac{E}{2}) = x$.

Subcase 2.2: $k > 1$. Then, $\sum_{p=0}^{k-1} c_1^p < E \leq \sum_{p=0}^k c_1^p$. The induction hypothesis is that for each $(c, E') \in \mathcal{C}^N$ with $\sum_{p=0}^{k-2} c_1^p < E' \leq \sum_{p=0}^{k-1} c_1^p$, $\varphi(c, E') = CEA(c, E')$. Let $E^{**} \equiv \sum_{p=0}^{k-1} c_1^p$. By *composition up*, we have $y = \varphi(c, E^{**}) + \varphi(c - \varphi(c, E^{**}), E - E^{**})$. Then, by the induction hypothesis, $\varphi(c, E^{**}) = CEA(c, E^{**}) = (\frac{E^{**}}{2}, \frac{E^{**}}{2})$. Note that $E - E^{**} \leq c_1^k$ and $\varphi_1(c, E^{**}) = \frac{E^{**}}{2} = c_1 - c_1^k$. Thus, $E - E^{**} \leq c_1 - \varphi_1(c, E^{**})$. Note that $c_1 - \varphi_1(c, E^{**}) \leq c_2 - \varphi_2(c, E^{**})$. By Case 1, $\varphi(c - \varphi(c, E^{**}), E - E^{**}) = (\frac{E - E^{**}}{2}, \frac{E - E^{**}}{2})$. It follows that $y = (\frac{E}{2}, \frac{E}{2}) = x$. Since *composition up* implies *resource continuity*, $\varphi(c, 2c_1) = CEA(c, 2c_1) = (c_1, c_1)$.

Case 3: $E > 2c_1$. Let $E^{***} \equiv 2c_1$. By Case 2, $\varphi(c, E^{***}) = (c_1, c_1)$. Thus, $c - \varphi(c, E^{***}) = (0, c_2 - c_1)$. By *composition up*, we have $y = \varphi(c, E^{***}) + \varphi((0, c_2 - c_1), E - E^{***})$. Note that $\varphi_1((0, c_2 - c_1), E - E^{***}) = 0$. It follows that $y_1 = c_1 = x_1$ and $y_2 = c_1 + E - E^{***} = E - c_1 = x_2$. Q.E.D.

For more than two creditors, the uniqueness part of Proposition 2 fails (A counterexample φ^* is provided in the next section). However, Proposition 2 extends naturally to the general case, if *consistency* comes to our rescue. Namely, the CEA rule is the only rule satisfying *secured lower bound*, *composition up*, and *consistency*. Moreover, as we show next, this characterization can be strengthened by weakening *consistency* to *null claims consistency*: the CEA rule is the only rule satisfying *secured lower bound*, *composition up*, and *null claims consistency*. The proof of the new characterization involves the following lemma.

Lemma 1 Let φ be a rule satisfying *secured lower bound* and *composition up*. For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ with $E \leq |N| \min_{i \in N} c_i$, $\varphi(c, E) = (\frac{E}{|N|}, \dots, \frac{E}{|N|})$.

Proof. Let φ be a rule satisfying *secured lower bound* and *composition up*. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ with $E \leq |N| \min_{i \in N} c_i$. Let n be the cardinality of N . Without loss of generality, we assume that $N \equiv \{1, \dots, n\}$ and $c_1 \leq \dots \leq c_n$. Let $y \equiv \varphi(c, E)$ and $x \equiv (\frac{E}{n}, \dots, \frac{E}{n})$. To show that $y = x$, we distinguish two cases.

Case 1: $E \leq c_1$. Since $c_1 \leq \dots \leq c_n$, then by *secured lower bound*, $y \geq (\frac{E}{n}, \dots, \frac{E}{n}) = x$. Note that $\sum_{i \in N} y_i = E$. Thus, $y = x$.

Case 2: $c_1 < E \leq nc_1$. Let $c_1^0 \equiv c_1$, $c_1^1 \equiv (\frac{n-1}{n})c_1^0$, $c_1^2 \equiv (\frac{n-1}{n})c_1^1$, and so on. Let $k \in \mathbb{N}$ be such that $k \leq n$ and $\sum_{p=0}^{k-1} c_1^p < E \leq \sum_{p=0}^k c_1^p$. The proof is by induction on k .

Subcase 2.1: $k = 1$. Let $E^* \equiv c_1^0$. By *composition up*, $y = \varphi(c, E^*) + \varphi(c - \varphi(c, E^*), E - E^*)$. By Case 1, $\varphi(c, E^*) = (\frac{E^*}{n}, \dots, \frac{E^*}{n})$. Note that $E - E^* \leq c_1^1$ and $c_1 - \varphi_1(c, E^*) = c_1^1$. It follows that $E - E^* \leq c_1 - \varphi_1(c, E^*)$. Note that $c_1 - \varphi_1(c, E^*) \leq c_2 - \varphi_2(c, E^*) \leq \dots \leq c_n - \varphi_n(c, E^*)$. By Case 1, $\varphi(c - \varphi(c, E^*), E - E^*) = (\frac{E-E^*}{n}, \dots, \frac{E-E^*}{n})$. Thus, $y = (\frac{E}{n}, \dots, \frac{E}{n}) = x$.

Subcase 2.2: $k > 1$. Then $\sum_{p=0}^{k-1} c_1^p < E \leq \sum_{p=0}^k c_1^p$. The induction hypothesis is that for each $(c, E') \in \mathcal{C}^N$ with $\sum_{p=0}^{k-2} c_1^p < E' \leq \sum_{p=0}^{k-1} c_1^p$, $\varphi(c, E') = (\frac{E'}{n}, \dots, \frac{E'}{n})$. We show that $y = x$. Let $E^{**} \equiv \sum_{p=0}^{k-1} c_1^p$. By *composition up*, $y = \varphi(c, E^{**}) + \varphi(c - \varphi(c, E^{**}), E - E^{**})$. The induction hypothesis gives us $\varphi(c, E^{**}) = (\frac{E^{**}}{n}, \dots, \frac{E^{**}}{n})$. Note that $E - E^{**} \leq c_1^k$ and $\varphi_1(c, E^{**}) = \frac{E^{**}}{n} = c_1 - c_1^k$. Thus, $E - E^{**} \leq c_1 - \varphi_1(c, E^{**})$. Note that $c_1 - \varphi_1(c, E^{**}) \leq c_2 - \varphi_2(c, E^{**}) \leq \dots \leq c_n - \varphi_n(c, E^{**})$. By Case 1, $\varphi(c - \varphi(c, E^{**}), E - E^{**}) = (\frac{E-E^{**}}{n}, \dots, \frac{E-E^{**}}{n})$. Thus, $x = (\frac{E}{n}, \dots, \frac{E}{n}) = y$. Q.E.D.

With the help of Lemma 1, we are now ready to prove the announced characterization of the CEA rule.¹³

Theorem 2 The CEA rule is the only rule satisfying *secured lower bound*, *composition up*, and *null claims consistency*.

Proof. Clearly, the CEA rule satisfies the three properties. Conversely, let φ be a rule satisfying the properties. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, $y \equiv \varphi(c, E)$, and

¹³For more characterizations of the constrained equal awards rule, readers are referred to Yeh (2004) and Yeh (2006).

$x \equiv CEA(c, E)$. Let n be the cardinality of N . Without loss of generality, we assume that $N \equiv \{1, \dots, n\}$ and $c_1 \leq \dots \leq c_n$. To show that $y = x$, we consider two cases.

Case 1: $E \leq nc_1$. By Lemma 1, $y = x$.

Case 2: $E > nc_1$. Let $c_0 \equiv 0$. Let $k \in \mathbb{N}$ be such that $k \leq n$ and $\sum_{p=0}^{k-1} c_p + (n - k + 1)c_k < E \leq \sum_{p=0}^k c_p + (n - k)c_{k+1}$. The proof is by induction on k .

Subcase 2.1: $k = 1$. Let $E^* \equiv nc_1$. Then $E^* < E \leq c_1 + (n - 1)c_2$. By *composition up*, $y = \varphi(c, E^*) + \varphi(c - \varphi(c, E^*), E - E^*)$. Let $c' \equiv c - \varphi(c, E^*)$. By Case 1, $\varphi(c, E^*) = (c_1, \dots, c_1)$. It follows that $c'_1 = 0$. By *null claims consistency*, $\varphi(c'_{N \setminus \{1\}}, E - E^*) = \varphi_{N \setminus \{1\}}(c', E - E^*)$. Thus, $\varphi_1(c', E - E^*) = 0$. Note that $E \leq c_1 + (n - 1)c_2$. It follows that $E - E^* \leq (n - 1)(c_2 - c_1)$. By Lemma 1, $\varphi(c'_{N \setminus \{1\}}, E - E^*) = (\frac{E - E^*}{n - 1}, \dots, \frac{E - E^*}{n - 1})$. It follows that $y_1 = c_1 = x_1$ and that for each $i \in N \setminus \{1\}$, $y_i = c_1 + \frac{E - E^*}{n - 1} = x_i$.

Subcase 2.2: $k > 1$. Then $\sum_{p=0}^{k-1} c_p + (n - k + 1)c_k < E \leq \sum_{p=0}^k c_p + (n - k)c_{k+1}$. The induction hypothesis is that for each $N \in \mathcal{N}$ and each $(c, E') \in \mathcal{C}^N$ with $\sum_{p=0}^{k-2} c_p + (n - k + 2)c_{k-1} < E' \leq \sum_{p=0}^{k-1} c_p + (n - k + 1)c_k$, $\varphi(c, E') = CEA(c, E')$. We show that $y = x$. Let $E^{**} \equiv \sum_{p=0}^{k-1} c_p + (n - k + 1)c_k$. By *composition up*, $y = \varphi(c, E^{**}) + \varphi(c - \varphi(c, E^{**}), E - E^{**})$. By the induction hypothesis, $\varphi(c, E^{**}) = CEA(c, E^{**})$. Let $c'' \equiv c - \varphi(c, E^{**})$ and $N' \equiv \{1, \dots, k\}$. Since $c_1 \leq \dots \leq c_n$, then for each $i \in N'$, $c''_i = 0$. Also, for each $i \in N \setminus N'$, $c_i - \varphi_i(c, E^{**}) = c_i - c_k$. By *null claims consistency*, $\varphi(c''_{N \setminus N'}, E - E^{**}) = \varphi_{N \setminus N'}(c'', E - E^{**})$. It follows that for each $i \in N'$, $\varphi_i(c'', E - E^{**}) = 0$. Note that $\sum_{p=0}^{k-1} c_p + (n - k + 1)c_k < E \leq \sum_{p=0}^k c_p + (n - k)c_{k+1}$ and $\varphi_{k+1}(c, E^{**}) = c_k$. Therefore, $E - E^{**} \leq (n - k)(c_{k+1} - \varphi_{k+1}(c, E^{**}))$. By Lemma 1, we have $\varphi(c''_{N \setminus N'}, E - E^{**}) = CEA(c''_{N \setminus N'}, E - E^{**})$. Since the CEA rule satisfies *composition up*, it follows that $y = x$. Q.E.D.

4 Discussion

We next show independence of the properties listed in each of our results, then derive the “dual” characterizations of our results, compare our results

with existing characterizations in the literature, and make a final remark.

4.1 Independence of the properties

To show independence of the properties in each of our characterizations, we introduce additional rules. The first rule assigns amounts such that the losses experienced by all creditors are equal subject to no one receiving a negative amount (e.g., Aumann and Maschler, 1985).

Constrained equal losses rule, CEL : For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$CEL_i(c, E) \equiv \max \{0, c_i - \lambda\},$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEL_i(c, E) = E$.

The random arrival rule (e.g., O'Neill, 1982) recommends the average of the awards vectors obtained by imagining creditors arriving one at a time and fully compensating them until there is nothing left, under the assumption that all orders of arrival are equally probable. Given $N \in \mathcal{N}$, let Π^N be the class of bijections on N .

Random arrival rule, RA : For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$RA_i(c, E) \equiv \frac{1}{|N|!} \sum_{\pi \in \Pi^N} \min \left\{ c_i, \max \left\{ E - \sum_{j \in N, \pi(j) < \pi(i)} c_j, 0 \right\} \right\}.$$

The rule φ^* mentioned in Section 3 is defined as follows. If the number of creditors is 3, there is only one creditor with the smallest claim, and the liquidation value is more than three times the smallest claim, then a “weighted constrained equal awards rule” is applied; otherwise, the CEA rule is applied. Formally, let $N \in \mathcal{N}$ with $|N| = 2$ and $\alpha \in \mathbb{R}_+^N$ with $\sum_{i \in N} \alpha_i = 1$.

Weighted constrained equal awards rule with weights α , CEA^α : For each $N \in \mathcal{N}$ with $|N| = 2$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$CEA_i^\alpha(c, E) \equiv \min \{c_i, \alpha_i \lambda\},$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i^\alpha(c, E) = E$.

Rule φ^* : For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$,

$$\varphi^*(c, E) \equiv \begin{cases} S(c, E) & \text{if } N \equiv \{i, j, k\}, c_i < \min\{c_j, c_k\} \text{ and } E > 3c_i; \\ CEA(c, E) & \text{otherwise,} \end{cases}$$

where S is defined as follows: Let $c_j^* \equiv c_j - c_i$, $c_k^* \equiv c_k - c_i$, $\alpha \equiv (\alpha_j, \alpha_k) \equiv (\frac{1}{3}, \frac{2}{3})$, and $E^* \equiv E - 3c_i$. Then, $S_i(c, E) \equiv c_i$ and $S_{N \setminus \{i\}}(c, E) \equiv (c_i, c_i) + CEA^\alpha(c_j^*, c_k^*; E^*)$

Property / Rule	<i>CEL</i>	<i>T</i>	<i>CEA</i>	<i>RA</i>	φ^*	<i>CD</i>
secured lower bound	No	Yes	Yes	Yes	Yes	Yes
minimal rights first	Yes	Yes	No	Yes	No	Yes
composition up	Yes	No	Yes	No	Yes	No
consistency	Yes	Yes	Yes	No	No	NA
null claims consistency	Yes	Yes	Yes	Yes	No	NA

Table 1: **Independence of the properties in each of our results.** The notation “Yes” (“No”) means that a certain rule satisfies (violates) a certain property. The notation “NA” means that a certain property is not applicable.

Table 1 shows that the properties listed in each of our results are logically independent. For instance, the CEA rule satisfies *secured lower bound* but violates *minimal rights first*. The CEL rule satisfies *minimal rights first* but violates *secured lower bound*. Thus, the properties listed in Proposition 1 are logically independent. The same is true for Proposition 2, and Theorems 1 and 2.

4.2 Dual results

We now derive dual characterizations of our results. Such characterizations are obtained by exploiting dual relations between rules, and between properties of rules. Given a rule φ , its dual, denoted by φ^d , is obtained by first replacing the liquidation value with its “complement” (the difference between the sum of the claims and the liquidation value), then applying φ to distribute that

difference, and finally subtracting the resulting awards vector from the claims vector. Formally, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\varphi^d(c, E) \equiv c - \varphi(c, \sum_{i \in N} c_i - E)$. We say that a rule is *self-dual* if it is dual to itself. Clearly, the Talmud rule and CD are *self-dual* (e.g., Aumann and Maschler, 1985). The CEA and CEL rules are dual to each other (e.g., Herrero and Villar, 2001).

Similarly, to any property can be associated a dual property. Two properties are dual if whenever a rule satisfies one of them, its dual satisfies the other. The dual of *secured lower bound* is *secured upper bound*: the loss experienced by each creditor should be at least as large as the minimum of his claim and the difference between the sum of all claims and the liquidation value divided by the number of creditors (e.g., Moreno-Tertero and Villar, 2004). The dual of *minimal rights first* is *invariance under claims truncation*: the awards vector chosen for the problem obtained by truncating creditors' claims at the liquidation value should be the same as it was initially (e.g., Curiel et al., 1987; Thomson, 2003). The dual of *composition up* is *composition down*: suppose that after an awards vector is chosen for a problem, the liquidation value is found to be less than initially thought. There are two ways to deal with this decrease. One is to cancel the initial division and recalculate the awards for the revised liquidation value. The other is to take the awards calculated on the basis of the initial liquidation value as claims in dividing the revised liquidation value. The requirement is that both ways of proceeding should yield the same awards vector (e.g., Thomson, 2003).¹⁴ Next are examples of self-dual properties. When looking at a bankruptcy problem, two perspectives can be taken. One can focus either on what is available or on what is missing (the difference between the sum of the claims and the liquidation value). *Self-duality* (e.g., Aumann and Maschler, 1985) says that both perspectives should be equivalent: what is available should be divided symmetrically to what is missing. Clearly, *self-duality* is a self-dual property. *Consistency* and *null claims consistency* are also examples of self-dual properties (e.g., Thomson and Yeh, 2007; Thomson, 2001a).

Thus, the dual of Proposition 1 says that in the two-creditor case, CD is the only rule satisfying *secured upper bound* and *invariance under claims truncation*. The dual of Theorem 1 says that the Talmud rule is the only

¹⁴Moulin (2000) refers to it as “upper composition.” This property is inspired by the notion of the property, “path independence,” which is proposed by Moulin (1987).

rule satisfying *secured upper bound*, *invariance under claims truncation*, and *consistency*. The dual of Proposition 2 says that in the two-creditor case, the CEL rule is the only rule satisfying *secured upper bound* and *composition down*. The dual of Theorem 2 says that the CEL rule is the only rule satisfying *secured upper bound*, *composition down*, and *null claims consistency*.

4.3 Comparison with existing results and final remark

We compare our results with existing characterizations in the literature. Dagan (1996) shows that in the two-creditor case, CD is the only rule satisfying *equal treatment of equals*: creditors with equal claims should receive equal amounts, *minimal rights first*, and *invariance under claims truncation*. Clearly, Proposition 1 is not implied by Dagan’s characterization. The “constrained egalitarian rule” (e.g., Chun, Schummer, and Thomson, 2001) satisfies *secured lower bound* and *equal treatment of equals* but violates *invariance under claims truncation*. The next rule satisfies *invariance under claims truncation* and *equal treatment of equals* but violates *secured lower bound*. If two claims are different, then a “sequential priority rule” (e.g., Moulin, 2000) is applied; otherwise, the CEA rule is applied.

Moreno-Tertero and Villar (2004) show that the Talmud rule is the only rule satisfying *secured lower bound*, *self-duality*, and *consistency*. The following rules show that Theorem 1 is not implied by their result. The “average rule” (e.g., Thomson, 2001b) satisfies *self-duality* but violates *minimal rights first*. The CEL rule satisfies *minimal rights first* but violates *self-duality*.

Moulin (2002) proposes another lower bound on each creditor’s award, *lower bound*: each creditor should receive at least the minimum of his claim and equal division. The author shows that the CEA rule is the only rule satisfying *lower bound*, *composition up*, and *consistency*. Our Theorem 2 strengthens Moulin’s result by weakening *lower bound* and *consistency* to *secured lower bound* and *null claims consistency*, respectively. This strengthening is quite significant since a number of well-known rules such as the random arrival and adjusted proportional rules satisfy *secured lower bound* and *null claims consistency*, but violate *lower bound* and *consistency*.

Finally, our Theorems 1 and 2 have the following implication. Note that *null claims consistency* is weaker than *consistency*. Thus, Theorem 2 implies

that the CEA rule is the only rule satisfying *secured lower bound*, *composition up*, and *consistency*. Surprisingly, this characterization and Theorem 1 differ only in the imposition of *minimal rights first* or *composition up*. Depending on which one of the two composition properties is imposed, we come up with quite different rules (the Talmud and the CEA rules). Thus, Theorems 1 and 2 provide insights into the fundamental differences between these rules.

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