Jang-Ting Guo Lecture 1-1

Introduction and Some Basics

The building blocks of modern macroeconomics are

(1) Solow (Neoclassical) growth model
Optimal (Ramsey) growth model
Real business cycle (RBC) model

(2) Overlapping generations (OLG) model

RBC	OLG
Finite #	Infinite #
Infinitely-Lived	Finite-Lived
CE $\Leftrightarrow$ PO	CE $\neq$ PO
Saddle	Sink
Determinate/Unique	Indeterminate/Multiple

Main Themes

(1) Macroeconomic models are described by non-linear stochastic difference/differential equations

(2) These models can be linearized around a steady state

(3) Examine the restrictions that macroeconomic theories places on the behavior of these equations, and then confront with the data

Common Characteristics

(1) Dynamic (Intertemporal)

(2) Microfoundation

(3) General Equilibrium

(4) Nonlinear $\Rightarrow$ Log-linearization

(5) Stochastic

(6) Rational Expectations

(7) Infinite Horizon Discrete/Continuous Time

(1) Dynamic $\Rightarrow$ U.S. Real GDP Per Person, 1890 - 2000

(2) Microfoundation

(A) Static Optimization

(a) over two consumption goods

$$\begin{aligned} &\text{Max } u(c_1, c_2) \\ &\text{s.t. } p_1c_1 + p_2c_2 = Y \end{aligned}$$

$$MRS_{1,2} = \frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

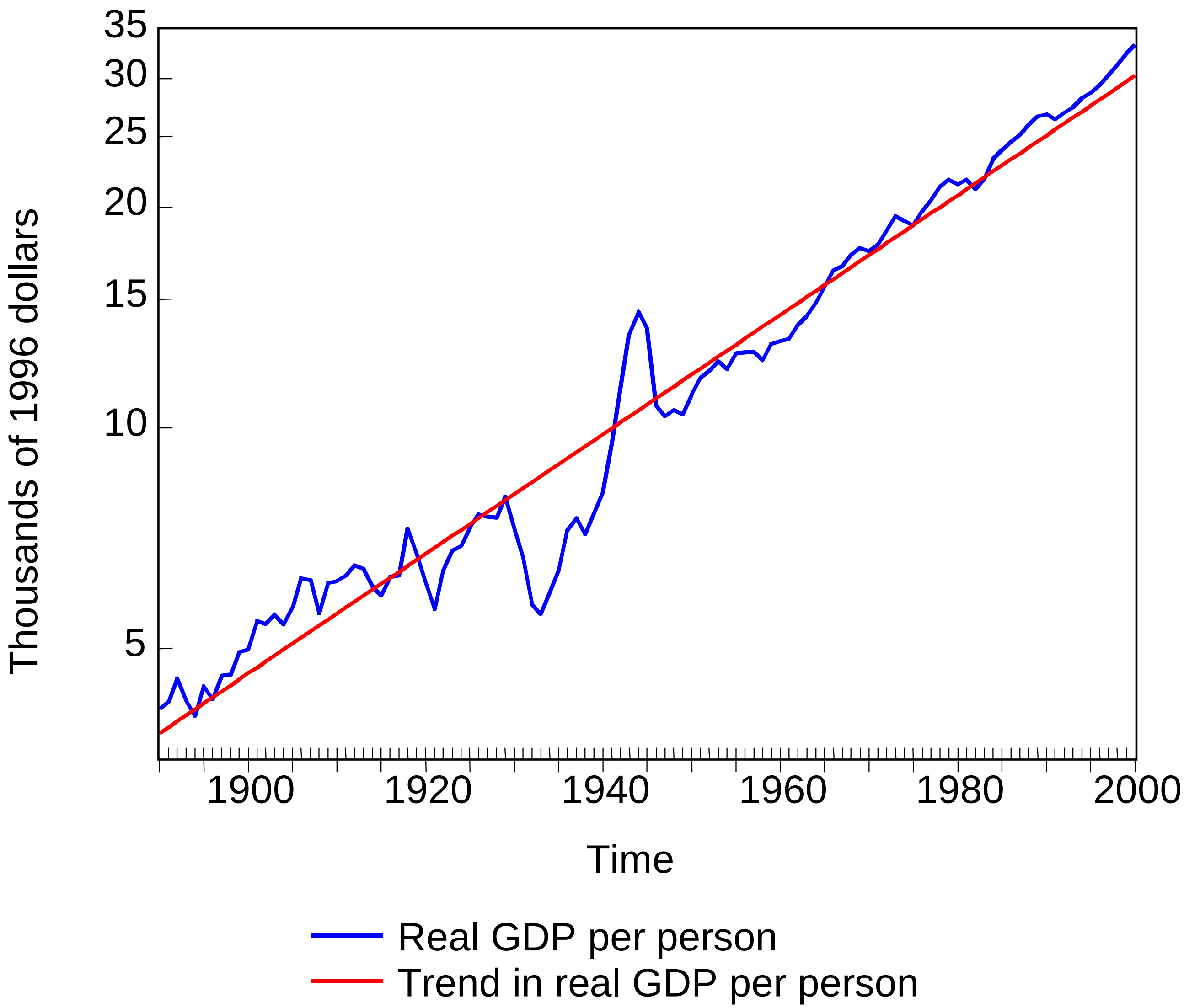
(b) over consumption and labor (leisure)

$$\begin{aligned} &\text{Max } u(c, \ell) \\ &\text{s.t. } pc = Wn = W(T - \ell) \\ &\Rightarrow \quad pc + W\ell = WT \end{aligned}$$

$$MRS_{\ell,c} = \frac{MU_\ell}{MU_c} = \frac{W}{p} = w$$

Since $\ell = T - n$, $MU_\ell = -MU_n$

$$MRS_{n,c} = -\frac{MU_n}{MU_c} = w$$



Real GDP Per Person in US, 1890-2000

(B) Dynamic Optimization

(a) one sector, two periods and endowment economy

$$\text{Max } u(c_t, c_{t+1})$$
$$\text{s.t. } p_t c_t + p_{t+1} c_{t+1} = p_t y_t + p_{t+1} y_{t+1}$$

$$MRS_{1,2} = \frac{MU_{c_t}}{MU_{c_{t+1}}} = \frac{p_t}{p_{t+1}} = 1 + r_{t+1}$$

(b) one sector, two periods and production economy

$$\text{Max } u(c_t) + \beta u(c_{t+1}),\ 0 < \beta < 1$$
$$\text{s.t. } c_t + k_{t+1} - (1-\delta)k_t = y_t = r_t k_t$$
$$y_t = f(k_t) \Rightarrow r_t = MPK_t$$
$$k_t \text{ given}$$

Next, form the Lagrangian

$$L = u(c_t) + \beta u(c_{t+1}) + \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right]$$

$$\frac{\partial L}{\partial c_t} = 0 \Rightarrow u'(c_t) = \lambda_t$$
$$\frac{\partial L}{\partial k_{t+1}} = 0 \Rightarrow \lambda_t = \beta \lambda_{t+1} [f'(k_{t+1}) + 1 - \delta]$$

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$$

(C) Putting things together

$$\text{Max } u(c_t, \ell_t) + \beta u(c_{t+1}, \ell_{t+1}), \; 0 < \beta < 1$$

$$\begin{aligned} \text{s.t.} \quad & c_t + k_{t+1} - (1-\delta)k_t = r_t k_t + w_t n_t \\ & y_t = f(k_t, n_t) \text{ and } k_t \text{ given} \\ & n_t + \ell_t = T \end{aligned}$$

(a) Intra-temporal FOC for labor supply

$$MRS_{n_t, c_t} = -\frac{MU_{n_t}}{MU_{c_t}} = w_t = MPN_t$$

(b) Inter-temporal FOC for consumption
Consumption-Euler Equation

$$u'(c_t) = \beta u'(c_{t+1})[r_{t+1} + 1 - \delta], \quad r_{t+1} = MPK_{t+1}$$

(3) General Equilibrium

Competitive Equilibrium
Pareto Optimum

First Welfare Theorem
Second Welfare Theorem

(4) Nonlinear $\Rightarrow$ Log-linearization

Consider $y_t = f(x_t)$

At the steady state $\bar{y} = f(\bar{x})$

Linearization in levels

$$(y_t - \bar{y}) = f'(\bar{x})(x_t - \bar{x})$$

Linearization in percentage deviations

$$\bar{y}\frac{(y_t - \bar{y})}{\bar{y}} = f'(\bar{x})\frac{(x_t - \bar{x})}{\bar{x}}\bar{x}$$

$$\hat{y}_t = \frac{f'(\bar{x})\bar{x}}{\bar{y}}\hat{x}_t \Rightarrow \hat{y}_t = \frac{f'(\bar{x})\bar{x}}{f(\bar{x})}\hat{x}_t$$

Log-Linearization

Use $\log(1+\hat{z}_t) \approx \hat{z}_t$ and $\hat{z}_t = \frac{z_t - \bar{z}}{\bar{z}}$

$\hat{y}_t = \log(y_t) - \log(\bar{y})$ and $\hat{x}_t = \log(x_t) - \log(\bar{x})$

$$\hat{y}_t = \frac{f'(\bar{x})\bar{x}}{f(\bar{x})}\hat{x} \Rightarrow \log(y_t) - \log(\bar{y}) = \frac{f'(\bar{x})\bar{x}}{f(\bar{x})}[\log(x_t) - \log(\bar{x})]$$

(5) Stochastic

Impulses: shocks to supply or demand

Propagation Mechanism: Exogenous or Endogenous

(6) Rational Expectations

General Formulation

(A) $y_t = f(y^e_{t+1})$, where y is an endogenous variable such as inflation rate or price level

(B) $y_t = f(x^e_{t+1})$, where y and x are both endogenous variables, e.g., y is consumption and x is income

Three Ways to Model y^e_{t+1}

(A) Perfect Foresight: $y^e_{t+1} = y_{t+1}$:

(B) Adaptive Expectations

$$y^e_{t+1} = y^e_t + \lambda(y_t - y^e_t), \text{ where } 0 < \lambda < 1$$
$$y^e_{t+1} = \lambda y_t + (1-\lambda) y^e_t$$

(C) Rational Expectations: $y^e_{t+1} = E[y_{t+1} \mid I_t]$

(7) Infinite Horizon Discrete/Continuous Time

Linear Rational Expectations Models

Consider a scalar rational expectations model

1) $y_t = \alpha E[y_{t+1}|\Omega_t] + x_t + v_t$

2) $x_t = \bar{x}$, for all t

3) $E_t[v_{t+s}] = 0$, for all $s > 0$

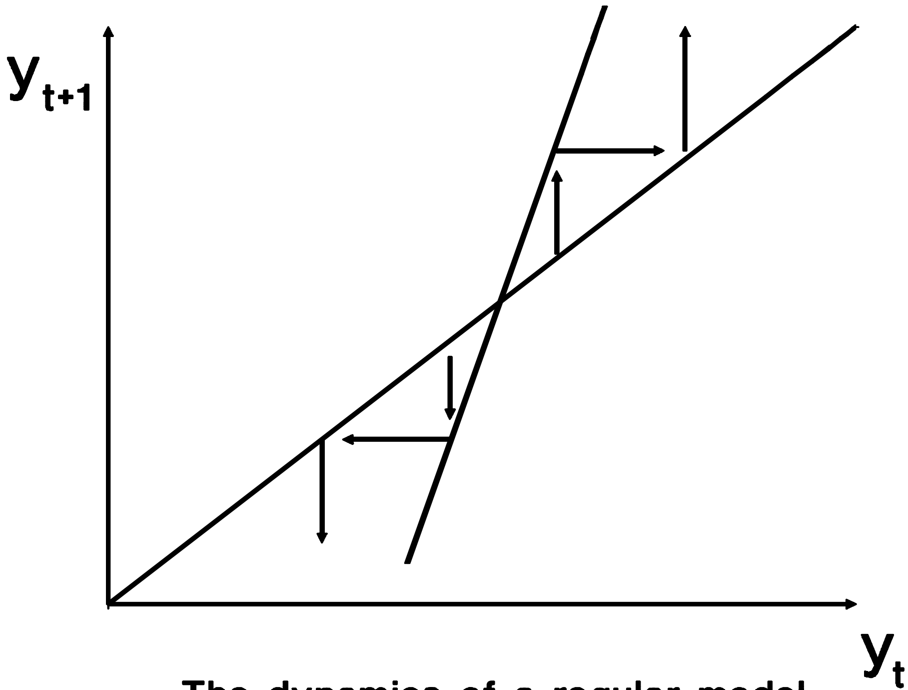
We will discuss two cases in which the solution remains bounded: $|y_t| < \infty$ for all t

Case 1: the REGULAR model where $0 < |\alpha| < 1$

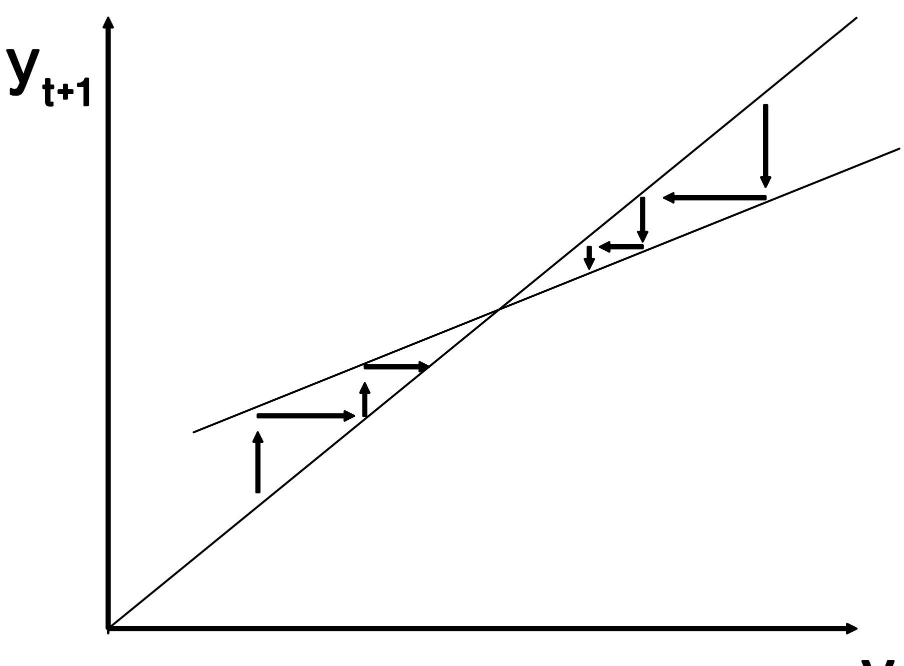
To solve the model, iterate equation (i) into the future:

4)
$$y_t = \bar{x} + v_t + \alpha E_t\{\bar{x} + v_{t+1} + \alpha E_{t+1}[\bar{x} + v_{t+2} + \ \ldots$$
$$= \frac{\bar{x}}{1-\alpha} + \sum_{s=0}^{\infty} \alpha^s E_t[v_{t+s}]$$

5) $y_t = \dfrac{\bar{x}}{1-\alpha} + v_t$



The dynamics of a regular model



The dynamics of an irregular model

Case 2: the IRREGULAR model where $|\alpha| > 1$

1) $y_t = \alpha E[y_{t+1}|\Omega_t] + x_t + v_t$

It is no longer possible to solve equation (i) forward.

But this does not mean that there is no rational expectations solution. In fact there are many!

Consider the following stochastic difference equation:

6) $y_{t+1} = -\frac{1}{\alpha}\overline{x} + \frac{1}{\alpha} y_t - \frac{1}{\alpha} v_t + \varepsilon_{t+1}$

where ε_{t+1} is a random variable with a time t conditional mean of zero, which can be interpreted as self-fulfilling beliefs of agents (sunspots, animal spirits)

By construction, (6) represents a solution to equation (1). Notice that since $|\alpha| > 1$, the solution remains bounded

Check: take conditional expectations on both sides of (6), we obtain that $E_t y_{t+1} = -\frac{1}{\alpha}\overline{x} + \frac{1}{\alpha} y_t - \frac{1}{\alpha} v_t$, which is exactly the same as equation (1)

Now, consider a <u>vector</u> rational expectations model

7) $$Y_t = AY_{t-1} + U_t$$

<u>Eigenvalues and Eigenvectors</u>

$$x \rightarrow Ax \qquad x \in R^n$$
$$Ax = \lambda x \qquad \lambda \text{ is a scalar}$$
$$|A - \lambda I| = 0 \text{ or } x = 0$$

Use n = 2 as an example where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

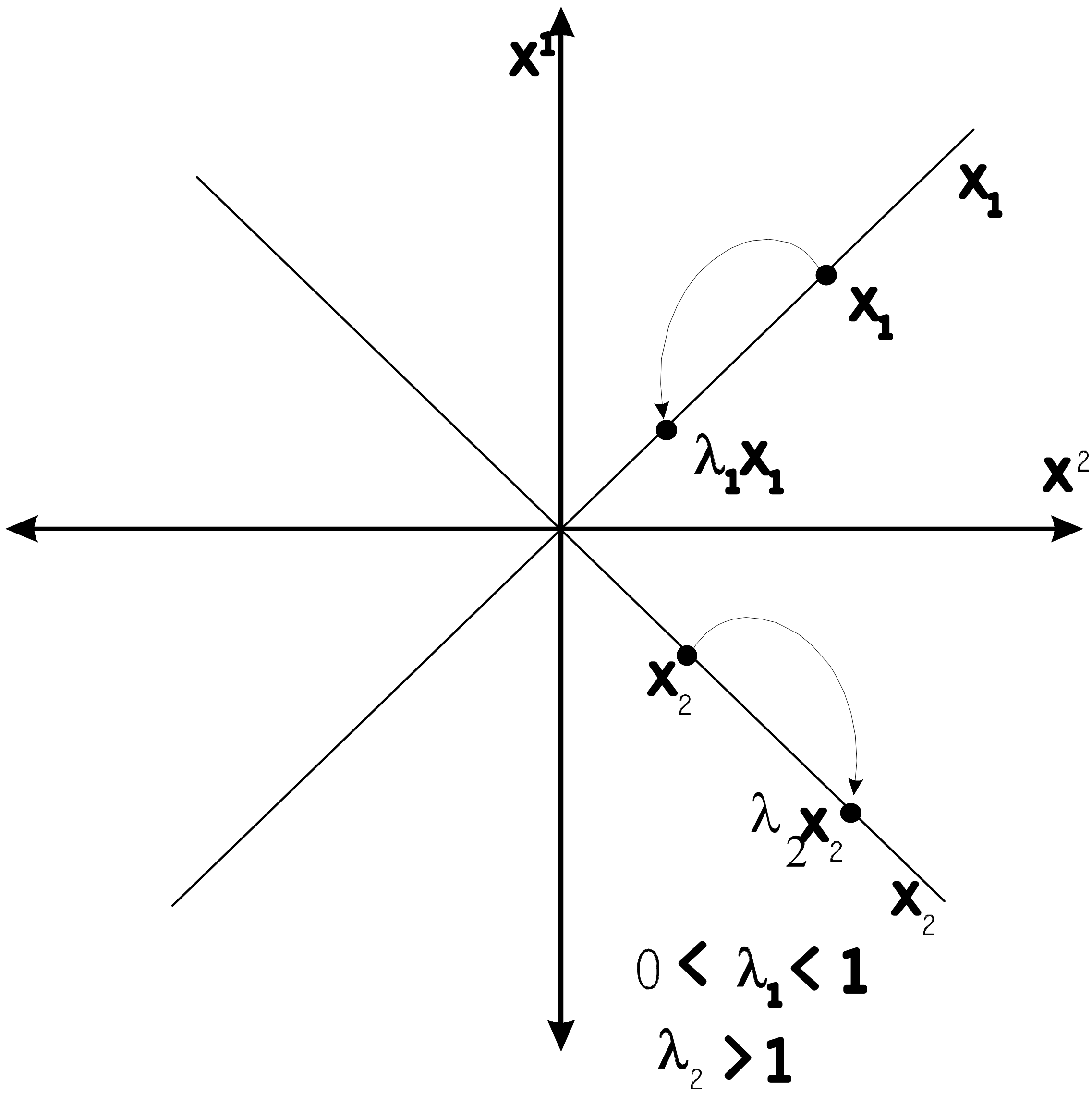
<u>Characteristic polynomial</u>

$$P(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$
$$P(\lambda) = \lambda^2 - (\text{tr}\,A)\lambda + \det A = 0$$

λ_1, λ_2 are <u>roots</u> or <u>eigenvalues</u> of the matrix A

$$\lambda_1 + \lambda_2 = \text{Trace}(A) = a_{11} + a_{22}$$
$$\lambda_1\lambda_2 = \text{Det}(A) = a_{11}a_{22} - a_{12}a_{21}$$

λ_i is <u>stable</u> if $|\lambda_i| < 1$ (inside the unit circle)
λ_i is <u>unstable</u> if $|\lambda_i| > 1$ (outside the unit circle)



One stable and one unstable eigenvalues

7) $$Y_t = AY_{t-1} + U_t$$

where $Y_t \in R^2$. Also $U_t \in U$ with bounded support, and a finite unconditional mean

Diagonalization of A

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1^1 & \vdots & y_2^1 \\ y_1^2 & \vdots & y_2^2 \end{pmatrix} = \begin{pmatrix} y_1^1 & \vdots & y_2^1 \\ y_1^2 & \vdots & y_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A(y_1, y_2) = (y_1, y_2)\Lambda, \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $Q = (y_1, y_2)$, and assume that Q is non-singular

$$AQ = Q\Lambda$$
$$A = Q\Lambda Q^{-1}$$

$$Y_t = AY_{t-1} = Q\Lambda Q^{-1}Y_{t-1}$$
$$Q^{-1}Y_t = \Lambda Q^{-1}Y_{t-1}$$

8)

$$\text{Let} \quad Z_t = \begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = Q^{-1} Y_t$$

$$\text{then} \quad Z_t = \Lambda Z_{t-1} + W_t$$

where $\begin{matrix} z_t^1 = \lambda_1 z_{t-1}^1 + w_t^1 \\ z_t^2 = \lambda_2 z_{t-1}^2 + w_t^2 \end{matrix}$ and $W_t = \begin{bmatrix} w_t^1 \\ w_t^2 \end{bmatrix} = Q^{-1} U_t$

where z_t^i evolve independently, i =1, 2. That is, we transform the original vector stochastic difference equation (7) into a system of two independent (uncoupled) stochastic scalar difference equations.

Stability of the steady state

Let Y* be the steady state of equation (7)

1) If all eigenvalues of A are inside the unit cycle, Y^* is said to be asymptotically stable and it is a sink

2) If at least one eigenvalues is outside the unit cycle, then Y^* is unstable. If this holds for all eigenvalues, Y* is a source, otherwise a saddle

3) If no eigenvalue of A is outside the unit cycle but at least one is on the boundary (has modulus 1), then Y^* maybe stable, asymptotically stable, or unstable. In this case, bifurcation will arise

An Example

$$9)\qquad \begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix} = A\begin{bmatrix} y_{t+1}^1 \\ E_t\left[y_{t+1}^2\right] \end{bmatrix} + Bx_t + \omega_t,\ y_0^1 \text{ given}$$

Let $A = Q\Lambda Q^{-1}$

Let

$$\begin{bmatrix} z_t^1 \\ z_t^2 \end{bmatrix} = Q^{-1}\begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix}$$

$$\begin{bmatrix} \phi_t^1 \\ \phi_t^2 \end{bmatrix} = Q^{-1}Bx_t$$

$$\begin{bmatrix} u_t^1 \\ u_t^2 \end{bmatrix} = Q^{-1}\omega_t$$

$$10)\qquad \begin{aligned} z_t^1 &= \lambda_1 E_t\left[z_{t+1}^1\right] + \phi_t^1 + u_t^1 \\ z_t^2 &= \lambda_2 E_t\left[z_{t+1}^2\right] + \phi_t^2 + u_t^2 \\ q_{21}z_0^1 + q_{22}z_0^2 &= \overline{y}_0^2 \end{aligned}$$

where $Q \equiv \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ = matrix of the eigenvectors.

For a <u>unique</u> rational expectations equilibrium

# of roots of A outside the unit circle <u>equals</u> # of predetermined initial conditions

In this 2x2 case with one predetermined variable this condition can be written as

$$|\lambda_1| < 1 < |\lambda_2| \Rightarrow \text{a saddle} \Rightarrow \text{Determinacy}$$

Solve the stable root forward to obtain the unique REE

11) $$z_t^1 = E_t \sum_{s=0}^{\infty} \lambda_1^s \left(\phi_{t+s}^1 + u_{t+s} \right)$$

For <u>multiple</u> rational expectations equilibria

# of roots of A outside the unit circle <u>greater than</u> # of predetermined initial conditions

In this 2x2 case with one predetermined variable,

$$1 < |\lambda_1| < |\lambda_2| \Rightarrow \text{a sink} \Rightarrow \text{Indeterminacy}$$

$\Rightarrow$ A continuum of stationary "sunspot" equilibria

Definition: Hyperbolic Equilibrium

Let $\bar{x}$ be a steady state of the non-linear dynamic system $x_{t+1} = f(x_t)$. $\bar{x}$ is hyperbolic if none of the eigenvalues for the Jacobian matrix of the partial derivatives $Df(\bar{x})$ falls on the unit circle. That is, no eigenvalue has modulus exactly equal to 1

Theorem: Hartmen-Grobman

Let $\bar{x}$ be a hyperbolic steady state of the non-linear dynamic system $x_{t+1} = f(x_t)$. There is a neighborhood U of $\bar{x}$ in which the non-linear dynamic system is topologically equivalent to the approximated linear system

$$x_{t+1} \approx \bar{x} + Df(\bar{x})(x_t - \bar{x})$$

Topologically equivalent systems exhibit the same qualitative dynamic properties

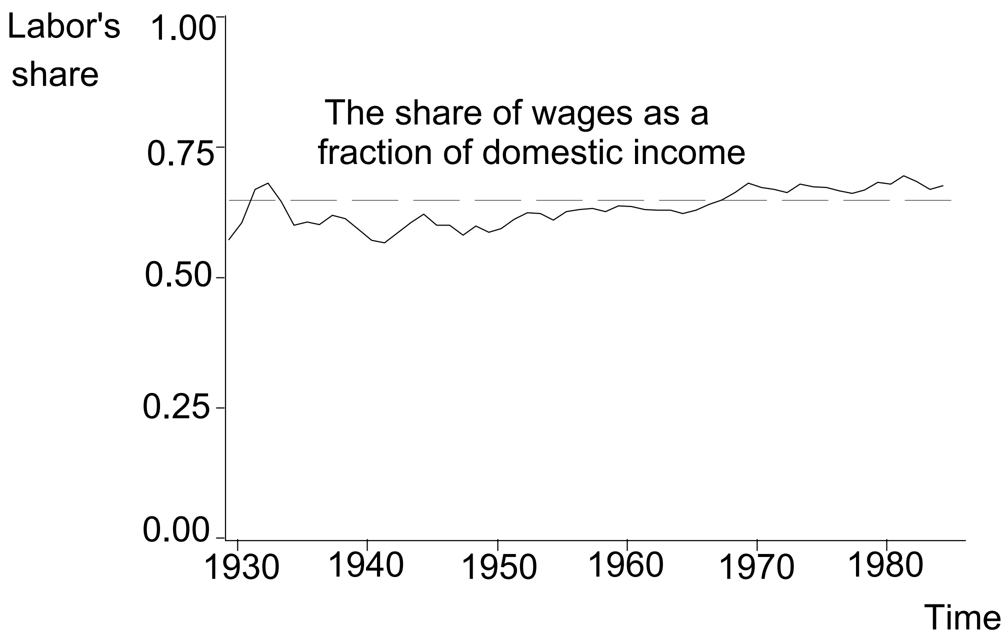
Local stability of a hyperbolic steady state $\bar{x}$ can be determined by the eigenvalues of the Jacobian matrix $Df(\bar{x})$, as in page 7

Jang-Ting Guo Lecture 1-3

Solow and Optimal Growth Models

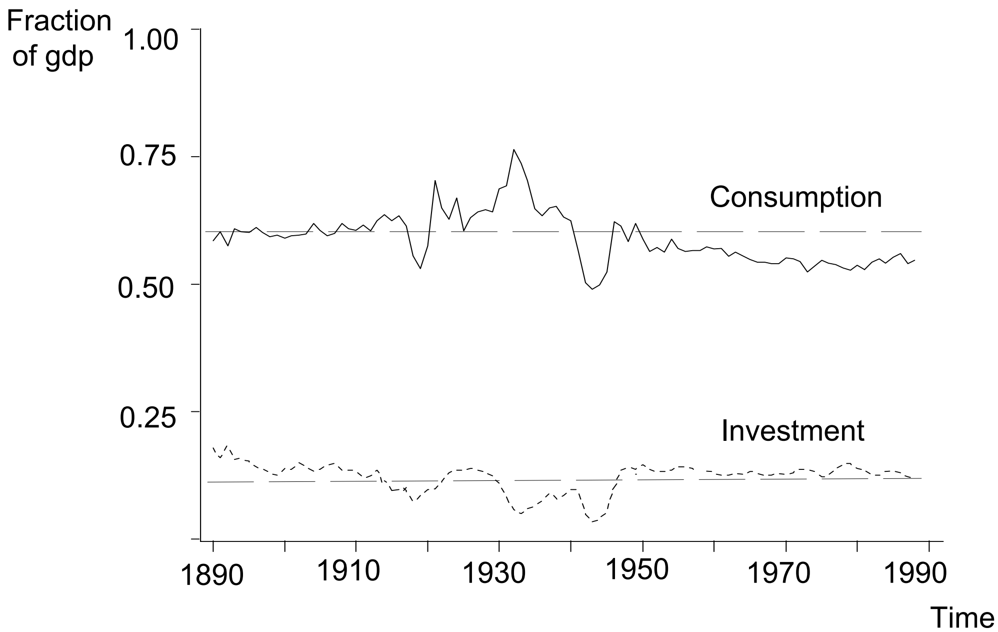
Three Stylized Facts about U.S. Economic Growth

(1) The wage share of national income has remained constant over long periods of time



Under the neoclassical assumption of constant return-to-scale, this fact implies that the production function must be Cobb-Douglas with elasticities of capital and labor that are equal to their respective factor shares in national income

(2) The ratios of consumption and investment to output are approximately constant in the data

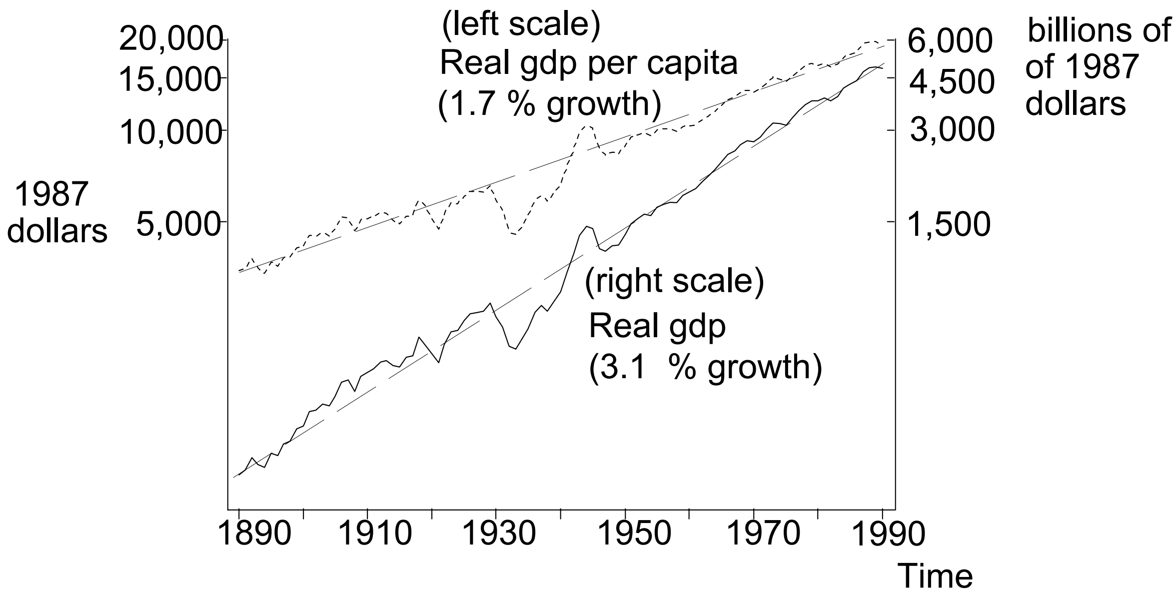


Since output, consumption and investment have all been growing over the past hundred years, this fact implies that output, consumption and investment have grown at <u>equal</u> rates ⇒ <u>growth is balanced</u>

(3) Real GDP and real GDP per capita (standard of living) have both grown at roughly a <u>constant</u> rate in the U.S. over the last hundred years

$$\frac{\Delta(Y/N)}{(Y/N)} = \frac{\Delta Y}{Y} - \frac{\Delta N}{N}$$

growth rate of GDP per person	=	growth rate of GDP	–	growth rate of population
1.7%	=	3.1%	–	1.4%



This fact implies that output growth has reached a steady state with a constant growth rate

Combining (2) and (3) implies that U.S. output, consumption and investment have grown at the same constant rate in the last century

The Solow growth model explains these facts by building them into the assumptions of the model:

(1) The constancy of labor share in national income is built into the assumption that output is produced with a Cobb-Douglas production function with the value of labor's and capital's share of 2/3 and 1/3, respectively

(2) The idea of balanced growth is built into the assumption of a constant savings rate so that

$$\frac{S}{Y} = s = \frac{I}{Y} \text{ and } \frac{C}{Y} = 1 - s$$

(3) The growth of real GDP per capita follows from an exogenous technical progress, that is, from the growth in productivity or the Solow residuals

Technology in the Solow Growth Model

The production function of output is

$$Y = K^{\alpha} L^{1-\alpha} S$$

Define efficiency units of labor as follows:

$$E^{1-\alpha} = L^{1-\alpha} S$$

Hence, the production function exhibits CRS in K and E

$$Y = K^{\alpha} E^{1-\alpha}$$

The Solow Growth Model without Technical Progress

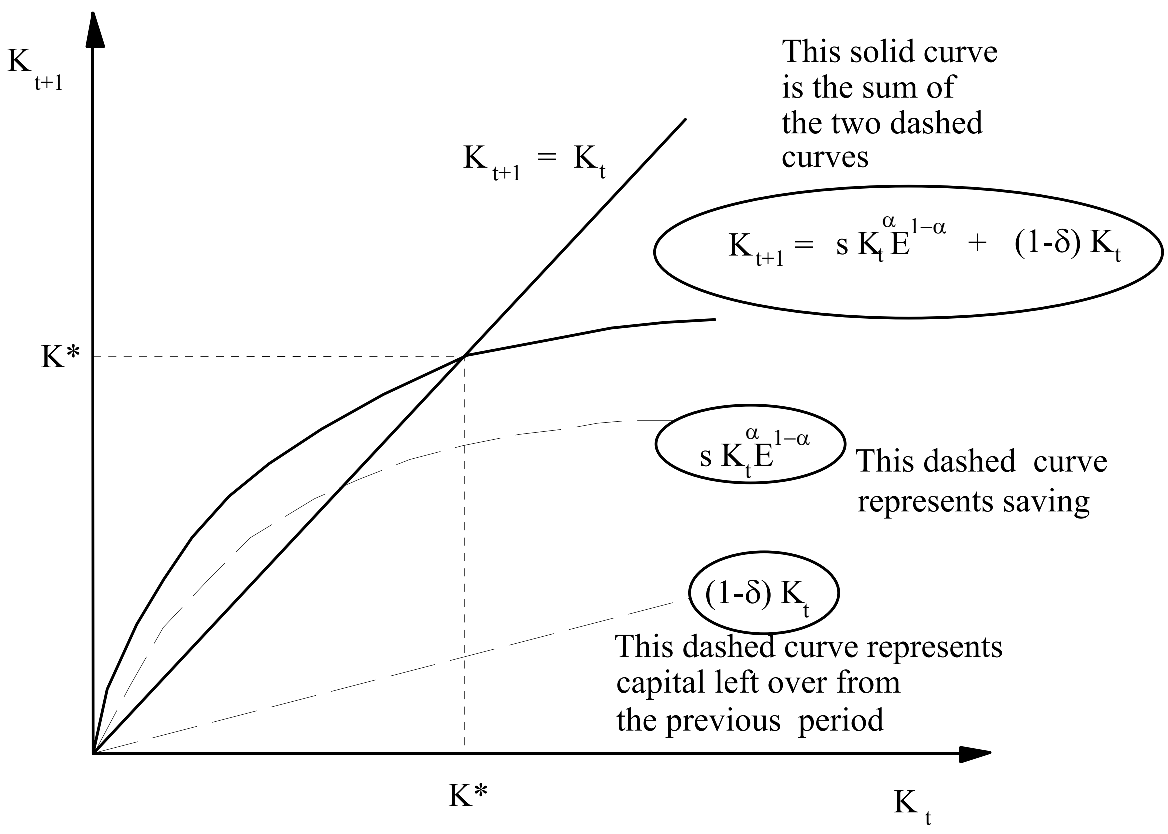
Assumptions

(1) In a closed economy, households save a fixed fraction (s) of their income $\Rightarrow I = S$

(2) There is no population growth and the state of technology (S) is kept unchanged

sY	=	s	x	$K_t^{\alpha} E_t^{1-\alpha}$	=	$K_{t+1} - (1-\delta)K_t$,
savings	=	savings rate	x	output	=	Investment

K_{t+1}	=	$(1-\delta)K_t$	+	$sK_t^{\alpha} E^{1-\alpha}$
next period's capital	=	capital left over after subtracting depreciation	+	a fraction "s" of output devoted to new investment



The Solow Growth Model with Technical Progress

Denote the rate of growth of efficiency units of labor as g. Then the economy can be described by

$$K_{t+1} = (1-\delta)K_t + sK_t^{\alpha}E_t^{1-\alpha}$$
$$E_{t+1} = (1+g)E_t$$

It follows that the steady-state ratio of K/E is given by

$$\left(\frac{K}{E}\right)^{*} = \left(\frac{s}{g+\delta}\right)^{\frac{1}{1-\alpha}}$$

Since E is increasing in every period and the fact that K/E converges to a constant implies that K must be increasing in every period by the same proportion as E. That is, growth in K and E are balanced

It follows that output and consumption per person at the steady state are given by:

$$\frac{Y^{*}}{N} = \left(\frac{K}{E}\right)^{*\alpha} S^{\frac{1}{1-\alpha}} \text{ and } \frac{C^{*}}{N} = (1-s)\frac{Y^{*}}{N}$$

With exogenous technical progress, per-capita output, capital and consumption all grow forever at the same constant rate as efficiency labor (=g)

Optimal Growth Model

1) $$\text{Max } E_0[\sum_{t=0}^{\infty}\beta^t \log c_t],\ 0<\beta<1,$$

2) $$c_t + k_{t+1} \leq (1-\delta)k_t + s_t k_t^{\alpha} n_t^{1-\alpha}$$

3) $n_t = 1$ (Inelastic labor supply)

4) $$k_0 = \bar{k}_0, \quad s_0 = \bar{s}_0$$

Next, form the Lagrangian

5) $$L = \sum_{t=0}^{\infty}\beta^t \{\log c_t + \lambda_t\left(s_t[k_t]^{\alpha} + (1-\delta)k_t - c_t - k_{t+1}\right)\}$$

First-order conditions

6) (i) $$\frac{1}{c_t} = \beta E_t\{\frac{1}{c_{t+1}}[\alpha s_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)]\}$$

(ii) $$k_{t+1} = (1-\delta)k_t + s_t k_t^{\alpha} - c_t$$

Steady State

Let $\bar{s} = E(s_t)$. The steady state $(\bar{k}, \bar{c})$ is the solution to the following system of equations

$$\frac{1}{\bar{c}} = \frac{\beta}{\bar{c}}\left[\alpha \bar{s}\bar{k}^{\alpha-1} + 1 - \delta\right]$$

$$\bar{k} = (1-\delta)\bar{k} + \bar{s}\bar{k}^{\alpha} - \bar{c}$$

Now, log-linearize 6) (i) and (ii) around the steady state

6)(i') $$a_1 \tilde{c}_t = E_t\left[a_2 \tilde{c}_{t+1} + a_3 \tilde{k}_{t+1} + a_4 \tilde{s}_{t+1}\right]$$

6)(ii') $$b_1 \tilde{k}_{t+1} = b_2 \tilde{k}_t + b_3 \tilde{s}_t - b_4 \tilde{c}_t$$

Regarding the technology shock s_t

$$s_t = s_{t-1}^{\rho} \nu_t, \quad 0 \le \rho < 1, \quad s_0 = \bar{s}_0,$$

$$\nu_t \text{ i.i.d. random variable with } E[\nu_t] = 1$$

$$\nu_t \in V = [\nu_1, \nu_2], \quad 0 < \nu_1 < \nu_2 < \infty, \quad t = 1, 2, \ldots$$

Notice that $X_t = X_{t+1} + \text{Expectations Error}$, and we can put these equations together as follows:

7)
$$\begin{bmatrix} a_1 & 0 & 0 \\ -b_4 & b_2 & b_3 \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} = \begin{bmatrix} a_2 & a_3 & a_4 \\ 0 & b_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\nu}_{t+1} \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix}$$

where $w^x_{t+1} = E_t[x_{t+1}] - x_{t+1}$, for $x = c,k,s$.

Pre-multiply (7) by the matrix $\begin{bmatrix} a_1 & 0 & 0 \\ -b_4 & b_2 & b_3 \\ 0 & 0 & \rho \end{bmatrix}^{-1}$, we obtain

8)
$$\begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} = A \begin{bmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{bmatrix} + B \begin{bmatrix} \tilde{\nu}_{t+1} \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix}, \quad \tilde{k}_0, \tilde{s}_0 \text{ given.}$$

Following the procedure we have discussed,

9)

$$y_t \equiv \begin{bmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{bmatrix} = Ay_{t+1} + Bu_{t+1}, \quad A = Q\Lambda Q^{-1}$$

$$Q^{-1}y_t = \Lambda Q^{-1}y_{t+1} + Q^{-1}Bu_{t+1}$$

Transform (9) into three independent scalar equations

10)

$$z_t \equiv Q^{-1}y_t = \Lambda z_{t+1} + \Phi_{t+1}, \quad \Phi_{t+1} = Q^{-1}Bu_{t+1}$$

$$\text{for row } i \quad z^i_t = \lambda^i z^i_{t+1} + \phi^i_{t+1} \quad i = 1,2,3.$$

In this 3x3 case with 2 predetermined variables, the condition for a <u>unique</u> rational expectations equilibrium is

$$|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$$

Since λ_1 is the stable root, solve it forward:

11)

$$z^1_t = \phi^1_{t+1} + \lambda^1\{\phi^1_{t+2} + \lambda^1[\phi^1_{t+3} + \ldots$$

Taking conditional expectations on both sides of (11) leads to $z_t^1 = 0$, for all t. It follows that

12)	$z_t^1 = q^{11}\tilde{c}_t + q^{12}\tilde{k}_t + q^{13}\tilde{s}_t = 0 \quad \text{for all t}$

where (q^{11}, q^{12}, q^{13}) forms the first row of Q^{-1}

Equation (12) defines a linear restriction on the vector y_t that relates the value of the free variable, c_t, to the values of the predetermined variables, k_t and s_t. That is, (12) describes the stable branch of the saddle path, which characterizes the unique rational expectations equilibrium