Moment-Based Copula Tests for Financial Returns

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ABSTRACT

In this paper, we propose a class of moment-based tests for copulae in a parametric multivariate dynamic context for financial returns. The proposed method takes into account the effect of estimation uncertainty. This effect is quite important but often ignored in related studies. Our method can be applied to generate various tests to detect copula mis-specification in different directions. In particular, on the basis of the conditional probabilities of quantile-exceedances (Kendall's tau), it generates the tail-dependence tests (the concordance test) that can be used to investigate whether the copula being tested is suitable for characterizing the true tail-dependence (concordance) structure. Such tests may be useful for exploring the cross-dependence structures of financial returns that are essential for risk management and other purposes. The Monte Carlo simulation supports the validity of our method. As a demonstrative application, we also apply these tests to an empirical study of stock market relationships.

KEY WORDS: Copula; Cross-dependence; Method of Moments; Specification Test.

1 INTRODUCTION

There is a rapidly growing interest in modelling the cross-dependence structures of financial returns by methods involving copula that deal with different problems, such as market relationships, Value-at-Risk (VaR), derivatives pricing, portfolio optimization, and financial contagion; see, e.g., Cherubini, Luciano, and Vecchiato (2004). The rising popularity of this approach may be explained by its flexibility in accommodating different return distributions and various cross-dependence structures. This flexibility permits researchers to be free of the classical scope of normality and linear correlation; see, e.g., Embrechts, Lindskog, and McNeil (2003) for its importance in financial economics. Nevertheless, there is a wide variety of parametric copulae that could imply quite different cross-dependence structures; see, e.g., Hutchinson and Lai (1990) and Joe (1997). To avoid the biased conclusions caused by copula mis-specification, we should check the adequacy of copula models in characterizing the true cross-dependence structures by formal statistical tests.

In empirical studies, researchers are used to choosing copulae by the Akaike information criterion or other similar criteria. Another popular approach is to evaluate copulae based on Rosenblatt's (1952) multivariate probability integral transformation (PIT) theorem. This approach is closely related to the issue of evaluating the (multivariate) conditional probability density models, studied by Diebold, Gunther, and Tay (1998) and Diebold, Hahn, and Tay (1999), among others. The PIT theorem implies that the derivatives of the true copula, taken with respect to its margins and evaluated at the true PIT of returns, must be $U(0,1)$-distributed. Accordingly, several studies apply the classical goodness-of-fit tests, such as the Kolmogorov-Smirnov test or Pearson's χ^2 test, to check this uniformity hypothesis or its variants; see, e.g., Klugman and Parsa (1999) and Breymann, Dias, and Embrechts (2003), among many others. However, these classical tests are designed for the simple hypotheses that contain no unknown parameters, and their practical applications may not be theoretically valid for the reasons described below.

It is well recognized that financial returns have the stylized fact of volatility clustering and stock returns may even have the leverage effect; see, e.g., Engle (1982), Bollerslev (1986), and Nelson (1991). This means that financial returns are unlikely to be dynamically independent. As such, the role of returns in the PIT should be replaced with the standardized residuals of certain properly specified GARCH-type models that characterize the dynamic dependence structures. The parametric copula-based multivariate dynamic (CMD) models introduced by Hu (2006), Jondeau and Rockinger (2006), and Patton (2006a,b) are established by considering this fact. Because the GARCH-type models and most parametric copulae include some unknown parameters, the practical use of the classical tests must be based on certain parameter estimates. In this situation, the hypotheses are composite, rather than simple, and the classical tests encounter Durbin's (1973) problem, which means that their test statistics are not asymptotically pivotal in the presence of estimation uncertainty; see, e.g., Khmaladze (1981) and Fermanian and Scaillet (2004). Consequently, it may not be theoretically adequate to test the copulae of financial returns using the classical tests.

Recently, Chen, Fan, and Patton (2004) contributed two density-estimates-based copula tests in a semi-parametric CMD context. Their tests are invariant to the substitution of estimates for parameters and hence free of the estimation uncertainty. The density-estimates-based tests are of unspecified alternative hypotheses and general power directions. This is a good property in terms of test consistency. Nonetheless, financial analysts may also be interested in specific structures of cross-dependence for certain reasons. For example, they may want to concentrate on the concordance structure for exploring the financial market co-movements in normal times, the tail-dependence structure for assessing the VaR of portfolios, and the correlation asymmetry for risk management. In such situations, it becomes more important to consider the copula tests that have specific power directions, rather than universal powers.

In this paper, we introduce a flexible class of moment-based copula tests in a generalized parametric CMD context. This class of tests takes into consideration the estimation uncertainty effect. By being based on different moment conditions, it can be applied to generate various copula tests with distinctive power directions. In particular, on the basis of Kendall's tau (the conditional probabilities of quantile-exceedances), it yields the concordance test (the tail-dependence tests) for checking the mis-specification of co-movement (tail-dependence). These tests may shed light on the possible sources of copula mis-specification. This property is not only important for the above-mentioned financial applications but is also essential for copula re-specification. A Monte Carlo simulation demonstrates the importance of correcting the estimation uncertainty effect in testing copula and provides evidence that supports the validity of our tests. In our empirical study, we apply the proposed test to explore stock market relationships. By using the daily returns of seven stock indices, we observe that the normal and t copulae evidently outperform the Gumbel and Gumbel-survival copulae.

The rest of this paper is organized as follows. In Section 2, we discuss the parametric CMD context and establish the proposed method. In Section 3, we demonstrate the applicability of the proposed method. Section 4 includes a Monte Carlo simulation. Section 5 contains an empirical study. Finally, we conclude the paper in Section 6. The Appendix presents a simple estimation method used in this study.

2 THE PROPOSED METHOD

Let $\boldsymbol{y}_t := (y_{1t}, y_{2t}, \ldots, y_{nt})^\top$ be an $n \times 1$ vector of continuous random variables at time t for some fixed n with "$\top$" denoting the operator of transpose, and I_i^{t-1} be an information set generated by $Y_i^{t-1} := (y_{i,t-1}, y_{i,t-2}, \ldots)$ and some pre-determined variables at time t with $i = 1, 2, \ldots, n$. Given the information set $\boldsymbol{I}^{t-1} := (I_1^{t-1}, I_2^{t-1}, \ldots, I_n^{t-1})$, the cross-dependence of $\boldsymbol{y}_t$ is fully characterized by the true conditional multivariate distribution of $\boldsymbol{y}_t|\boldsymbol{I}^{t-1}$, denoted as $F_{\boldsymbol{y}}^o(\cdot|\boldsymbol{I}^{t-1})$. Let $F_{y_i}^o(\cdot|\boldsymbol{I}^{t-1})$ be the true conditional distribution of $y_{it}|\boldsymbol{I}^{t-1}$. The conditional Sklar theorem by Patton (2006a) indicates that there exists a unique conditional copula $C_o(\cdot|\boldsymbol{I}^{t-1}) : [0,1]^n \to [0,1]$ such that

$$F_{\boldsymbol{y}}^o(\boldsymbol{y}|\boldsymbol{I}^{t-1}) = C_o\left(F_{y_1}^o(y_1|\boldsymbol{I}^{t-1}), F_{y_2}^o(y_2|\boldsymbol{I}^{t-1}), \ldots, F_{y_n}^o(y_n|\boldsymbol{I}^{t-1})\middle|\, \boldsymbol{I}^{t-1}\right), \tag{1}$$

for all $\boldsymbol{y} := (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. This demonstrates that we may establish a parametric CMD model for $F_{\boldsymbol{y}}^o(\cdot|\boldsymbol{I}^{t-1})$ by coupling the marginal models for the $F_{y_i}^o(\cdot|\boldsymbol{I}^{t-1})$'s with the copula model for $C_o(\cdot|\boldsymbol{I}^{t-1})$.

To establish the marginal models, we consider the following multivariate framework:

$$\boldsymbol{y}_t = \boldsymbol{m}_t(\boldsymbol{x}_t, \boldsymbol{\alpha}) + \boldsymbol{h}_t(\boldsymbol{x}_t, \boldsymbol{\alpha})^{1/2}\boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\alpha} \in \boldsymbol{A} \subset \mathbb{R}^a, \tag{2}$$

where $\boldsymbol{x}_t$ is a vector of $\boldsymbol{I}^{t-1}$-measurable random variables; $\boldsymbol{m}_t := \boldsymbol{m}_t(\boldsymbol{x}_t, \boldsymbol{\alpha})$ is a $n \times 1$ vector with the ith element $m_{it} := m_{it}(\boldsymbol{x}_t, \alpha_i)$; $\boldsymbol{h}_t := \boldsymbol{h}_t(\boldsymbol{x}_t, \boldsymbol{\alpha})$ is a $n \times n$ diagonal matrix with the ith diagonal term $h_{it} := h_{it}(\boldsymbol{x}_t, \alpha_i)$; $\boldsymbol{\alpha} := (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_n^\top)^\top$ is an $a \times 1$ parameters vector in the parameters space $\boldsymbol{A}$, $\alpha_i \in A_i \subset \mathbb{R}^{a_i}$ is an $a_i \times 1$ parameters vector, and $a = \sum_{i=1}^n a_i$; $\boldsymbol{\varepsilon}_t := (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{nt})^\top$ is the standardized errors vector with $\varepsilon_{it} := h_{it}^{-1/2}(y_{it} - m_{it})$, $\mathbb{E}[\varepsilon_{it}] = 0$, and $\text{var}[\varepsilon_{it}] = 1$, and $\varepsilon_{it}|\boldsymbol{x}_t$ has the conditional distribution $F_{\varepsilon_i}(\cdot|\boldsymbol{x}_t; \beta_i)$ with a $b_i \times 1$ parameters vector $\beta_i \in B_i \subset \mathbb{R}^{b_i}$ and the conditional probability density function:

$$f_{\varepsilon_i}(\epsilon|\boldsymbol{x}_t; \beta_i) := \frac{\partial}{\partial \epsilon} F_{\varepsilon_i}(\epsilon|\boldsymbol{x}_t; \beta_i), \qquad \forall\, \epsilon \in \mathbb{R};$$

we also denote $\boldsymbol{\beta} := (\beta_1^\top, \beta_2^\top, \ldots, \beta_n^\top)^\top \in \boldsymbol{B} \subset \mathbb{R}^b$ and $b := \sum_{i=1}^n b_i$. This generates the marginal models:

$$F_{y_i}(y|\boldsymbol{x}_t; \gamma_i) := F_{\varepsilon_i}\left(h_{it}^{-1/2}(y - m_{it})\middle|\, \boldsymbol{x}_t; \beta_i\right), \qquad \forall\, y \in \mathbb{R}, \tag{3}$$

with the parameters vector $\gamma_i := (\alpha_i^\top, \beta_i^\top)^\top \in \Gamma_i \subset \mathbb{R}^{a_i + b_i}$; $i = 1, 2, \ldots, n$.

Let $C(\cdot|\boldsymbol{x}_t; \theta)$ be a copula model, being used to approximate the true conditional copula $C_o(\cdot|\boldsymbol{I}^{t-1})$, that has the parameters vector $\theta \in \Theta \subset \mathbb{R}^r$. By coupling the marginal models with this copula model, a generalized parametric CMD model is derived:

$$F_{\boldsymbol{y}}(\boldsymbol{y}|\boldsymbol{x}_t; \lambda) := C\left(F_{y_1}(y_1|\boldsymbol{x}_t; \gamma_1), F_{y_2}(y_2|\boldsymbol{x}_t; \gamma_2), \ldots, F_{y_n}(y_n|\boldsymbol{x}_t; \gamma_n)\middle|\, \boldsymbol{x}_t; \theta\right), \tag{4}$$

where $\lambda := (\boldsymbol{\gamma}^\top, \theta^\top)^\top$ is a $(a+b+r)\times 1$ vector of parameters with $\boldsymbol{\gamma} := (\boldsymbol{\alpha}^\top, \boldsymbol{\beta}^\top)^\top$. This context encompasses the copula-based models of Hu (2006), Jondeau and Rockinger (2006), and Patton (2006a,b). The constant conditional correlation (CCC) model of Bollerslev (1990) and the dynamic conditional correlation (DCC) models of Engle (2002) and Tse and Tsui (2002) are also encompassed by this context and correspond to the case of conditional multivariate normality where the F_{y_i}'s and C are both normal.

The key feature of this context is that the parameter vectors γ_i's are separable for different i's. This permits us to present the marginal models as a set of univariate GARCH-type models conditional on the same information set $\boldsymbol{I}^{t-1}$. Accordingly, we can estimate the γ_i's separately before the copula analysis. As demonstrated by Bauwens, Laurent, and Rombouts (2006, Section 2.3), the CCC (or DCC) model and the copula-based models of Jondeau and Rockinger (2006) and Patton (2006a) are in the same sub-class of multivariate GARCH-type models obtained by certain nonlinear combinations of univariate GARCH-type models. This interpretation applies to the generalized CMD model (4). The VEC model of Bollerslev, Engle, and Wooldridge (1988) and the BEKK model of Engle and Kroner (1995) are other types of multivariate GARCH-type models that may not have completely separable parameters for various i's. This difference makes the CCC and DCC models much easier to estimate than the VEC and BEKK models; see, e.g., Engle and Sheppard (2001), Engle (2002), and Tse and Tsui (2002). In addition to this advantage, the CMD model can also be flexibly applied to explore the cross-dependence structures using various C's.

The aim of this study is to propose a class of moment-based tests for copulae in the context of (4). To focus on testing the copula model, we make the following assumption:

[A] The marginal models are correctly specified in the sense that there exists some unique vector $\gamma_{io} := (\alpha_{io}^\top, \beta_{io}^\top)^\top \in \Gamma_i$ at which

$$F_{y_i}(\cdot|\boldsymbol{x}_t;\gamma_{io}) = F_{y_i}^o(\cdot|\boldsymbol{I}^{t-1}), \qquad \forall\ i = 1, 2, \ldots, n.$$

By comparing (1) with (4), it is clear that, given assumption [A], the model $F_{\boldsymbol{y}}(\cdot|\boldsymbol{x}_t;\lambda)$ is correctly specified for the true conditional multivariate distribution $F_{\boldsymbol{y}}^o(\cdot|\boldsymbol{I}^{t-1})$ if the null hypothesis:

$$H_o\ :\ C(\cdot|\boldsymbol{x}_t;\theta_o) = C_o(\cdot|\boldsymbol{I}^{t-1}), \tag{5}$$

for some unique $\theta_o \in \Theta$, is satisfied.

Denote $u_{it} := F_{y_i}(y_{it}|\boldsymbol{x}_t;\gamma_i)$ and $\boldsymbol{u}_t := (u_{1t}, u_{2t}\ldots, u_{nt})^\top$. Given assumption [A], $u_{it}^o := u_{it}|_{\gamma_i=\gamma_{io}}$ is the true conditional PIT of $y_{it}|\boldsymbol{I}^{t-1}$, and hence $\boldsymbol{u}_{ot} := (u_{1t}^o, u_{2t}^o, \ldots, u_{nt}^o)^\top$

is a $n \times 1$ vector of $U(0,1)$ random variables. Let $\phi_t := \phi(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)$ be a $q \times 1$ vector of testing indicators. Denote $\gamma_o := (\boldsymbol{\alpha}_o^\top, \boldsymbol{\beta}_o^\top)^\top$, $\lambda_o := (\boldsymbol{\gamma}_o^\top, \theta_o^\top)^\top$, and $\phi_{ot} := \phi_t|_{\lambda=\lambda_o}$. Suppose that the condition $\mathbb{E}[\phi_{ot}|\boldsymbol{I}^{t-1}] = 0$ holds under the null hypothesis. Accordingly, we can check the null hypothesis by examining a simple testable implication: $\mathbb{E}[\phi_{ot}] = 0$. (Some practical and useful examples will be proposed in Section 3.2.)

Given assumption [A], let $\hat{\alpha}_{iT}$ and $\hat{\beta}_{iT}$ be, respectively, certain $\sqrt{T}$-consistent estimators of α_{io} and β_{io}, and $\hat{\theta}_T$ be a $\sqrt{T}$-consistent estimator of θ_o under the null hypothesis. Denote the standardized residuals $\hat{\varepsilon}_{it} := \varepsilon_{it}|_{\alpha_i=\hat{\alpha}_{iT}}$, $\hat{\boldsymbol{\alpha}}_T := (\hat{\alpha}_{1T}^\top, \hat{\alpha}_{2T}^\top, \ldots, \hat{\alpha}_{nT}^\top)^\top$, $\hat{\boldsymbol{\beta}}_T := (\hat{\beta}_{1T}^\top, \hat{\beta}_{2T}^\top, \ldots, \hat{\beta}_{nT}^\top)^\top$, $\hat{\gamma}_{iT} := (\hat{\alpha}_{iT}^\top, \hat{\beta}_{iT}^\top)^\top$, $\hat{\boldsymbol{\gamma}}_T := (\hat{\boldsymbol{\alpha}}_T^\top, \hat{\boldsymbol{\beta}}_T^\top)^\top$, $\hat{\lambda}_T := (\hat{\boldsymbol{\gamma}}_T^\top, \hat{\theta}_T^\top)^\top$, $\hat{\boldsymbol{u}}_t := \boldsymbol{u}_t|_{\boldsymbol{\gamma}=\hat{\boldsymbol{\gamma}}_T}$, and $\hat{\phi}_t := \phi_t|_{\lambda=\hat{\lambda}_T}$. We will base our test on the following statistic:

$$\hat{D}_T := T^{-1} \sum_{t=1}^{T} \hat{\phi}_t,$$

and establish the asymptotic null distribution of $\sqrt{T}\hat{D}_T$ using the generalized first-order asymptotics of Phillips (1991) to take into account the effect of estimation uncertainty.

Let ∇_{α_i}, ∇_{β_i}, and ∇_θ be the partial derivative operators taken with respect to α_i, β_i, and θ, respectively. Denote $p_{it} := \frac{\partial}{\partial u_{it}}\phi_t$, $w_{it} := (\nabla_{\alpha_i} m_{it}) h_{it}^{-1/2}$, $z_{it} := (\nabla_{\alpha_i} h_{it}) h_{it}^{-1}$, $f_{it} := f_{\varepsilon_i}(\varepsilon_{it}|\boldsymbol{x}_t;\beta_i)$, and $\mathcal{F}_{it} := \nabla_{\beta_i^\top} F_{\varepsilon_i}(\varepsilon_{it}|\boldsymbol{x}_t;\beta_i)$. Note that $\nabla_{\alpha_i^\top} u_{it} = -f_{it}\left(w_{it}^\top + \frac{1}{2} z_{it}^\top \varepsilon_{it}\right)$ and $\nabla_{\beta_i^\top} u_{it} = \mathcal{F}_{it}$. If ϕ is twice continuously differentiable, then we can apply the standard Taylor expansion to show that

$$\begin{aligned}\sqrt{T}\hat{D}_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi_{ot} + \left[\frac{1}{T}\sum_{t=1}^{T} \nabla_{\theta^\top}\phi_t\right]_{\lambda=\lambda_o} \sqrt{T}(\hat{\theta}_T - \theta_o) \\ + \sum_{i=1}^{n} \left\{ \left[\frac{1}{T}\sum_{t=1}^{T} p_{it} \nabla_{\gamma_i^\top} u_{it}\right]_{\lambda=\lambda_o} \sqrt{T}(\hat{\gamma}_{iT} - \gamma_{io}) \right\} + o_p(1),\end{aligned} \tag{6}$$

in which $\nabla_{\gamma_i^\top} u_{it} := (\nabla_{\alpha_i^\top} u_{it}, \nabla_{\beta_i^\top} u_{it})$; the "$o_p(1)$" term is due to the $\sqrt{T}$-consistency of estimators and a certain uniform law of large numbers that are needed as regularity conditions for the standard first-order asymptotics; see, e.g., Davidson (1994) and White (2001), among many others.

By using a suitable law of large numbers, we may further show that

$$\frac{1}{T}\sum_{t=1}^{T} \nabla_{\theta^\top}\phi_t \bigg|_{\lambda=\lambda_o} \xrightarrow{p} \eta_{co} := \mathbb{E}[\nabla_{\theta^\top}\phi_t]_{\lambda=\lambda_o},$$

and

$$\frac{1}{T}\sum_{t=1}^{T} p_{it}\nabla_{\gamma_i^\top} u_{it}\Bigg|_{\lambda=\lambda_o} \xrightarrow{p} \eta_{io} := \left[-\left\{\mathbb{E}[p_{it}f_{it}w_{it}^\top] + \frac{1}{2}\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top]\right\} \quad \mathbb{E}\left[p_{it}\mathcal{F}_{it}\right]\right]_{\lambda=\lambda_o},$$

in which $\mathbb{E}[p_{it}f_{it}w_{it}^\top] = \mathbb{E}[p_{it}f_{it}]\mathbb{E}[w_{it}^\top]$ and $\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top] = \mathbb{E}[p_{it}f_{it}\varepsilon_{it}]\mathbb{E}[z_{it}^\top]$ if ε_{it} is shown to be, or assumed to be, independent of $\boldsymbol{x}_t$. Consequently, we can re-express (6) as

$$\sqrt{T}\hat{D}_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_{ot} + \eta_{co}\sqrt{T}(\hat{\theta}_T - \theta_o) + \sum_{i=1}^{n}\left\{\eta_{io}\sqrt{T}(\hat{\gamma}_{iT} - \gamma_{io})\right\} + o_p(1). \qquad (7)$$

This demonstrates that, because of the estimation uncertainty, the asymptotic null distribution of $\sqrt{T}\hat{D}_T$ will not be free of the asymptotic distributions of $\sqrt{T}(\hat{\boldsymbol{\gamma}}_T - \boldsymbol{\gamma}_o)$ and $\sqrt{T}(\hat{\theta}_T - \theta_o)$ in general. The theoretical inadequacy of testing copulae using the classical tests discussed before arises due to ignoring such an effect.

This asymptotic method is standard for twice continuously differentiable testing indicators. By using the "generalized function" approach that Phillips (1991) introduced to construct the asymptotic normality of the least absolute deviation estimator, it may also be extended to the ϕ's composed of the indicator function: $I(\epsilon \geq \epsilon_o) = 1$ if $\epsilon \geq \epsilon_o$ and $I(\epsilon \geq \epsilon_o) = 0$ if $\epsilon < \epsilon_o$, where $\epsilon, \epsilon_o \in \mathbb{R}$. The validity of (7) for such ϕ's may be justified by the arguments of Phillips (1991, p.453-455) and will be supported by our Monte Carlo simulation. This is due to the fact that, although the indicator function is not differentiable in the ordinary sense, it is "differentiable" in terms of generalized functions and has the "generalized derivative":

$$\frac{\partial}{\partial\epsilon}I(\epsilon \geq \epsilon_o) = \delta(\epsilon - \epsilon_o),$$

where δ represents the Dirac delta function (or the so-called impulse symbol).

The Dirac delta function is a generalized function that can be understood as the limit of a delta sequence, such as the limit of the $N(\epsilon_o, \sigma^2)$ probability density functions sequence as $\sigma^2 \to 0^+$. Interestingly, this generalized function is known to have the sifting property (or said the reproducing property):

$$\int_{\mathbb{R}} \delta(\epsilon - \epsilon_o)\mu(\epsilon)\mathrm{d}\epsilon = \mu(\epsilon_o), \qquad (8)$$

where μ denotes a "test function" for linear functionals of δ; see Gelfand and Shilov (1964), Bracewell (1999), and Kanwal (2004), among others. By using this property and the definition of expectation, we will be able to establish the indicator-function-based tests that are free of δ in their practical applications; see Section 3.2 for more details.

To accomplish the asymptotic null distribution of $\sqrt{T}\hat{D}_T$, we consider the estimation methods that have the following properties:

$$\sqrt{T}(\hat{\alpha}_{iT} - \alpha_{io}) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi^o_{\alpha,it} + o_p(1), \tag{9}$$

$$\sqrt{T}(\hat{\beta}_{iT} - \beta_{io}) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi^o_{\beta,it} + o_p(1), \tag{10}$$

and

$$\sqrt{T}(\hat{\theta}_T - \theta_o) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi^o_{\theta t} + o_p(1), \tag{11}$$

where $\{\psi^o_{\alpha,it}\}$, $\{\psi^o_{\beta,it}\}$, and $\{\psi^o_{\theta t}\}$ are certain martingale difference sequences such that $\mathbb{E}[\psi^o_{\alpha,it}|\boldsymbol{I}^{t-1}] = 0$ and $\mathbb{E}[\psi^o_{\beta,it}|\boldsymbol{I}^{t-1}] = 0$ under assumption [A] and $\mathbb{E}[\psi^o_{\theta t}|\boldsymbol{I}^{t-1}] = 0$ under assumption [A] and the null hypothesis.

Theoretically, we may estimate the parameter vectors λ_o by maximizing the likelihood function of $F_{\boldsymbol{y}}(\cdot|\boldsymbol{x}_t;\lambda)$ directly. The resulting maximum likelihood estimators (MLEs) are $\sqrt{T}$-consistent and asymptotically efficient under assumption [A] and the null hypothesis. However, this one-stage method is not necessarily easy to implement when the marginal and copula models are complicated. In practice, it would be much easier to estimate λ_o using certain multi-stage estimation methods. For this purpose, Patton (2006b) proposed a useful two-stage estimation method that first estimates γ_{io} by maximizing the likelihood function of the marginal model $F_{y_i}(\cdot|\boldsymbol{x}_t;\gamma_i)$ for all the i's, and then estimates the copula parameters θ_o by maximizing the $\hat{\boldsymbol{\gamma}}_T$-based likelihood function of $F_{\boldsymbol{y}}(\cdot|\boldsymbol{x}_t;(\hat{\boldsymbol{\gamma}}_T^\top,\theta^\top)^\top)$.

In the Appendix, we summarize the formulae for $\psi^o_{\alpha,it}$, $\psi^o_{\beta,it}$, and $\psi^o_{\theta t}$ obtained from a minor variation to the two-stage method. Specifically, before estimating θ_o, this method first estimates α_{io} using the Gaussian quasi-ML (QML) method for all the i's, and then estimates β_{io} by maximizing the $\hat{\alpha}_{iT}$-based likelihood function of $F_{y_i}(\cdot|\boldsymbol{x}_t;(\hat{\alpha}_{iT}^\top,\beta_i^\top)^\top)$ for all the i's. Clearly, this three-stage method is not considered for the estimation efficiency because the resulting $\hat{\gamma}_{iT}$ may be less efficient than that of the two-stage method in the case of conditional non-normality if the marginal models are correctly specified. Instead, this method is motivated by the fact that it could make the estimation of the marginal models even easier. Moreover, the Gaussian QML method is useful for estimating and testing the partially specified models: m_{it}'s and h_{it}'s in a robust way before analyzing the F_{ε_i}'s. This "bottom-up" procedure is important for obtaining the suitable standardized residuals to build the fully-specified (marginal) models; see, e.g., Wooldridge (1990, 1991) and Bollerslev and Wooldridge (1992) for more discussions. Nonetheless, it should be noted that the M test is applicable to all the estimation methods with the properties: (9), (10), and (11), including this three-stage method and the above-mentioned one-stage and two-stage methods.

By introducing (9), (10), and (11) into (7), we can obtain the following transformation:

$$\sqrt{T}\hat{D}_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varphi_{ot} + o_p(1), \tag{12}$$

where $\varphi_{ot} := \phi_{ot} + \eta_{co}\psi_{\theta t}^{o} + \sum_{i=1}^{n}\eta_{io}\psi_{\gamma,it}^{o}$ with $\psi_{\gamma,it}^{o} := (\psi_{\alpha,it}^{o\top}, \psi_{\beta,it}^{o\top})^{\top}$. Recall that the condition $\mathbb{E}[\phi_{ot}|\boldsymbol{I}^{t-1}] = 0$ holds under the null hypothesis, so that $\{\varphi_{ot}\}$ is a martingale difference sequence such that $\mathbb{E}[\varphi_{ot}|\boldsymbol{I}^{t-1}] = 0$ under assumption [A] and the null hypothesis. By using the martingale-difference central limit theorem and the Cramér-Wold device, we can obtain the asymptotic null distribution of $\sqrt{T}\hat{D}_T$:

$$\sqrt{T}\hat{D}_T \xrightarrow{d} N(0, \Omega_o) \tag{13}$$

with the asymptotic variance-covariance matrix $\Omega_o := \mathbb{E}[\varphi_{ot}\varphi_{ot}^{\top}]$.

Because this result has taken into account the effect of estimation uncertainty, Ω_o is more complicated than $\mathbb{E}[\phi_{ot}\phi_{ot}^{\top}]$, i.e. the asymptotic variance-covariance matrix of $T^{-1/2}\sum_{t=1}^{T}\phi_{ot}$, that ignores this effect. Nonetheless, we may still easily estimate Ω_o by using a simple outer-product estimator:

$$\hat{\Omega}_T := \frac{1}{T}\sum_{t=1}^{T}\hat{\varphi}_t\hat{\varphi}_t^{\top}, \quad \hat{\varphi}_t := \hat{\phi}_t + \hat{\eta}_{cT}\hat{\psi}_{\theta t} + \sum_{i=1}^{n}\hat{\eta}_{iT}\hat{\psi}_{\gamma,it}, \tag{14}$$

in which $\hat{\psi}_{\gamma,it} := (\hat{\psi}_{\alpha,it}^{\top}, \hat{\psi}_{\beta,it}^{\top})^{\top}$, and $\hat{\eta}_{cT}$, $\hat{\eta}_{iT}$, $\hat{\psi}_{\theta t}$, $\hat{\psi}_{\alpha,it}$, and $\hat{\psi}_{\beta,it}$ are, respectively, the $\hat{\lambda}_T$-based sample counterparts (or other consistent estimators) of η_{co}, η_{io}, $\psi_{\theta t}^{o}$, $\psi_{\alpha,it}^{o}$, and $\psi_{\beta,it}^{o}$. The consistency of $\hat{\Omega}_T$ for Ω_o can be justified by using the generalized first-order asymptotics and the $\sqrt{T}$-consistency of $\hat{\lambda}_T$.

Under the condition that Ω_o and $\hat{\Omega}_T$ are non-singular, we can define the test statistic:

$$M_T := T\hat{D}_T^{\top}\hat{\Omega}_T^{-1}\hat{D}_T$$

that has the standard asymptotic null distribution:

$$M_T \xrightarrow{d} \chi^2(q),$$

as implied by (13). Hereafter, we will refer to the M_T-based test as the M test. In the case where $q = 1$, we can also express the M test statistic as $M_T' = \sqrt{T}\hat{D}_T/\hat{\Omega}_T^{1/2}$. This statistic has the asymptotic null distribution $N(0,1)$, and its sign may contain some useful information about the discrepancy between the true and postulated cross-dependence structures. This moment-based test can check various types of the copula mis-specifications by choosing suitable ϕ's, as we will discuss in Section 3.2.

3 APPLICABILITY

In this section, we first review some representative bivariate copulae ($n = 2$) and their cross-dependence structures, and then apply the M test and suitable ϕ's to establish the concordance test and the tail-dependence tests on the basis of this discussion.

3.1 Copulae and Cross-Dependence

For notational brevity, our discussion will mainly focus on the static copulae. Following Jondeau and Rockinger (2006) and Patton (2006a), the results can be easily extended to the dynamic copulae by re-specifying the parameters of the static copulae as certain dynamic functions of $\boldsymbol{x}_t$. The details will be discussed later.

Sklar's (1959) theorem indicates that for a continuous bivariate random variable with the joint distribution $\boldsymbol{G} : \mathbb{R}^2 \to [0,1]$ and the marginal distributions $G_i : \mathbb{R} \to [0,1]$, $i = 1,2$, there exists a unique copula C such that

$$\boldsymbol{G}(v_1, v_2) = C(G_1(v_1), G_2(v_2)), \qquad \forall\ (v_1, v_2) \in \mathbb{R}^2. \tag{15}$$

Given the PITs: $u_1 = G_1(v_1)$ and $u_2 = G_2(v_2)$, this result can be re-expressed as

$$C(u_1, u_2) = \boldsymbol{G}(G_1^{-1}(u_1), G_2^{-1}(u_2)), \qquad \forall\ (u_1, u_2) \in [0,1]^2, \tag{16}$$

where G_i^{-1} is the quantile function of G_i, $i = 1,2$. Note that C has the same parameters vector as $\boldsymbol{G}$ and the C-survival copula is defined as

$$C^s(u_1, u_2) := u_1 + u_2 - 1 + C(1-u_1, 1-u_2), \qquad \forall\ (u_1, u_2) \in [0,1]^2. \tag{17}$$

Let $\boldsymbol{g}$ and g_i be the probability density functions of $\boldsymbol{G}$ and G_i, respectively. The copula C has the density function:

$$c(u_1, u_2) = \frac{\boldsymbol{g}(G_1^{-1}(u_1), G_2^{-1}(u_2))}{g_1(G_1^{-1}(u_1))g_2(G_2^{-1}(u_2))}, \qquad \forall\ (u_1, u_2) \in [0,1]^2. \tag{18}$$

Formula (16) can be viewed as a general form of parametric copulae depending on the choice of $\boldsymbol{G}$.

If $\boldsymbol{G}$ is a distribution of two independent random variables:

$$\boldsymbol{G}(v_1, v_2) = G_1(v_1)G_2(v_2),$$

then (16) becomes the independent copula:

$$C_I(u_1, u_2) := u_1 u_2.$$

If $\boldsymbol{G}$ is the standardized bivariate normal distribution with the correlation coefficient $\rho \in (-1, 1)$, then (16) generates the normal copula:

$$C_N(u_1, u_2; \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{v_1^2 - 2\rho v_1 v_2 + v_2^2}{2(1-\rho^2)}\right) \mathrm{d}v_2 \mathrm{d}v_1,$$

where Φ^{-1} is the quantile function of $N(0,1)$. If $\boldsymbol{G}$ is Gumbel's type-B bivariate extreme value distribution with the parameter $\vartheta \in (0, 1]$, then (16) yields the Gumbel copula:

$$C_G(u_1, u_2; \vartheta) = \exp\left[-\left((-\ln u_1)^{\frac{1}{\vartheta}} + (-\ln u_2)^{\frac{1}{\vartheta}}\right)^{\vartheta}\right].$$

Given C_G, we can also define the Gumbel-survival copula:

$$C_G^s(u_1, u_2; \vartheta_s) = u_1 + u_2 - 1 + C_G(1 - u_1, 1 - u_2; \vartheta_s)$$

that has the parameter $\vartheta_s \in (0, 1]$. If $\boldsymbol{G}$ is the bivariate t distribution with the parameter $\rho \in (-1, 1)$ and the degrees of freedom ν, then (16) becomes the t copula:

$$C_t(u_1, u_2; \rho, \nu) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi\nu\sqrt{1-\rho^2}} \left(1 + \frac{v_1^2 - 2\rho v_1 v_2 + v_2^2}{\nu(1-\rho^2)}\right)^{-\left(1+\frac{\nu}{2}\right)} \mathrm{d}v_2 \mathrm{d}v_1,$$

where t_ν^{-1} is the univariate Student's t quantile function with the degrees of freedom ν, and ρ is the correlation coefficient if $\nu > 2$. The t copula reduces to C_N as $\nu \to \infty$. The normal and Gumbel (Gumbel-survival) copulae degenerate to C_I, which implies no cross-dependence, as $\rho = 0$ and $\vartheta = 1$ ($\vartheta_s = 1$). Besides the case of C_I, the copulae C_N, C_G, C_G^s, and C_t may imply different cross-dependence structures, as discussed below.

A pair of uniform random variables is said to be concordant (dis-concordant) if their observations tend to cluster around the 45^o (-45^o) line: $u_1 = u_2$ ($u_1 = 1 - u_2$). In the copula literature, it is common to measure concordance (or dis-concordance) by using Kendall's tau:

$$\tau = 4 \iint_{[0,1]^2} C(u_1, u_2) \mathrm{d}C(u_1, u_2) - 1. \tag{19}$$

This measure is always bound in [-1,1]. Its sign represents the direction of concordance (positive for concordance and negative for dis-concordance), and its magnitude indicates the strength of concordance (or dis-concordance); see, e.g., Nelsen (1999). It is quite easy to see that C_I implies $\tau = 0$ (no concordance). It is also known that for the copulae C_N and C_t,

$$\tau = \frac{2}{\pi} \arcsin(\rho) \tag{20}$$

is a monotone transformation of ρ; see, e.g., Fang, Fang, and Kotz (2002); for C_G,

$$\tau = 1 - \vartheta$$

must be non-negative. Therefore, unlike the normal and t copulae, C_G and C_G^s are unable to interpret the structure of dis-concordance.

Let c_N, c_G, and c_G^s be, respectively, the copula density functions of C_N, C_G, and C_G^s. We also define the lower-u tail events by a set of conditional quantile-exceedances:

$$A_{iL}(u) := \left\{ y_{it} \,\middle|\, y_{it} < F_{y_i}^{o-1}(u|\boldsymbol{I}^{t-1}) \right\}, \qquad u \in (0, 0.5],$$

and the upper-u tail events by another set of conditional quantile-exceedances:

$$A_{iU}(u) := \left\{ y_{it} \,\middle|\, y_{it} \geq F_{y_i}^{o-1}(u|\boldsymbol{I}^{t-1}) \right\}, \qquad u \in [0.5, 1),$$

$i = 1, 2$. Note that c_N, c_G, and c_G^s can be derived in accordance with (18), and $A_{iL}(u) = \{u_{it}^o \mid u_{it}^o < u\}$ and $A_{iU}(u) = \{u_{it}^o \mid u_{it}^o \geq u\}$ hold under assumption [A]. To further compare the implied cross-dependence structures of C_N and C_G, we plot c_N and c_G with $\tau = 0.2, 0.7$ in Figure 1, and summarize the main features of this figure as follows.

First, C_N and C_G both have a higher density at the 45^o line. This reflects the concordance implied by positive τ's. Second, $c_N(u, u; \rho)$ and $c_G(u, u; \vartheta)$ both increase with the magnitude of $|u - 0.5|$. This means that the clustering tendency of the lower-u (upper-u) tail events increases as $u \to 0^+$ ($u \to 1^-$). Third, this clustering tendency increases with the strength of concordance. Fourth, $c_N(u, u; \rho)$ is symmetric to $u = 0.5$, but $c_G(u, u; \vartheta)$ is asymmetric to $u = 0.5$ and has a heavier upper tail. Therefore, these two copulae imply quite different tail properties. In accordance with the shape of c_N and c_G at the 45^o line, Hu (2006) referred to C_N and C_G as copulae with the "U-shaped" and "J-shaped" dependence structures, respectively. In addition, because $c_G^s(u_1, u_2; \vartheta_s) = c_G(1 - u_1, 1 - u_2; \vartheta_s)$, $\forall\ (u_1, u_2) \in [0, 1]$, c_G^s is mirror-symmetric to c_G about the line: $u_1 = 1 - u_2$ when $\vartheta = \vartheta_s$. By this mirror-symmetry, it should be understood that C_G^s is a copula with the "L-shaped" dependence structure (heavier lower tail). Similar to C_N, the t copula also has the U-shaped dependence structure. These terminologies are useful to reflect the dissimilarities between the tail properties implied by these copulae.

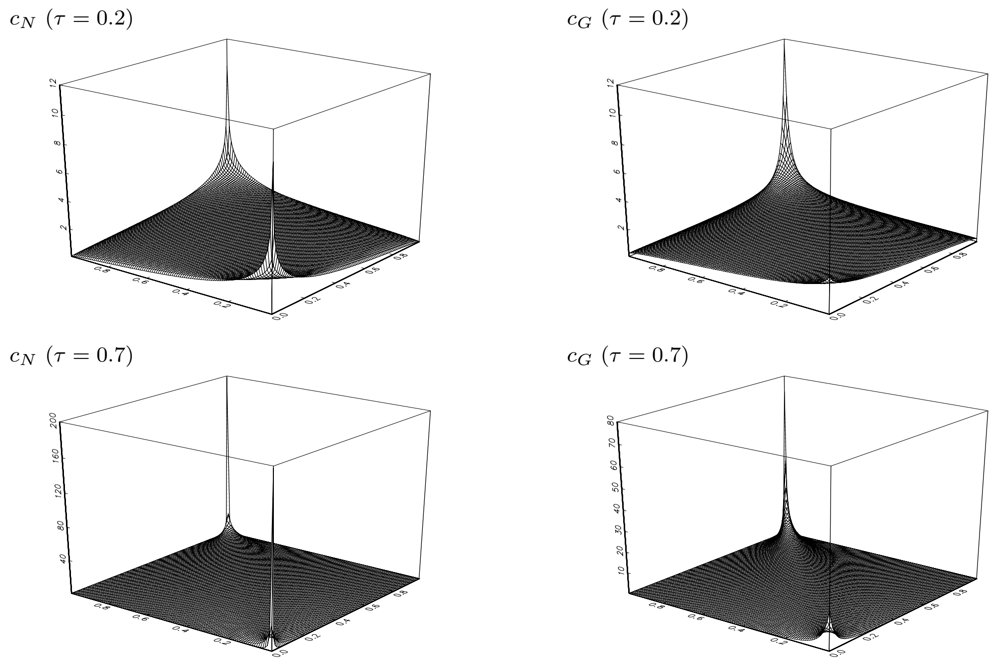

Figure 1. The normal and Gumbel copula density functions.

To characterize such tail properties more formally, we also define the lower-u tail-dependence measure by the conditional probability:

$$\lambda_L(u) := \mathbb{P}(A_{1L}(u)|A_{2L}(u)) = \frac{C_o(u,u)}{u}, \quad u \in (0, 0.5],$$

and the upper-u tail-dependence measure by the conditional probability:

$$\lambda_U(u) := \mathbb{P}(A_{1U}(u)|A_{2U}(u)) = \frac{C_o^s(1-u, 1-u)}{1-u}, \qquad u \in [0.5, 1),$$

where C_o^s is the survival copula of C_o, and C_o is temporarily assumed to be static, under assumption [A]. Note that $C^s(1-u, 1-u) = 1-2u+C(u,u) = \int_u^1 \int_u^1 c(u_1, u_2)\mathrm{d}u_2\mathrm{d}u_1$. It is easy to see that $\lambda_L(u)$ and $\lambda_U(u)$ are invariant to the replacements of $A_{2L}(u)|A_{1L}(u)$ and $A_{2U}(u)|A_{1U}(u)$, respectively, and they are bounded in [0,1] by the definition of probability. In fact, the Fréchet-Hoeffding inequality implies that the ratios: $C(u,u)/u$ and $C^s(1-u, 1-u)/(1-u)$ are always bounded in [0,1] for any copula C.

Given these definitions, the lower-u tail events are independent if, and only if, $\lambda_L(u) = u$; that is, $\mathbb{P}(A_{1L}(u)|A_{2L}(u)) = \mathbb{P}(A_{1L}(u))$. On the other hand, the upper-u tail events are independent if, and only if, $\lambda_U(u) = 1-u$. By contrast, the inequality $\lambda_L(u) \neq u$ ($\lambda_U(u) \neq 1-u$) implies the dependence of lower-u (upper-u) tail events. Clearly, $C_o = C_I$ implies

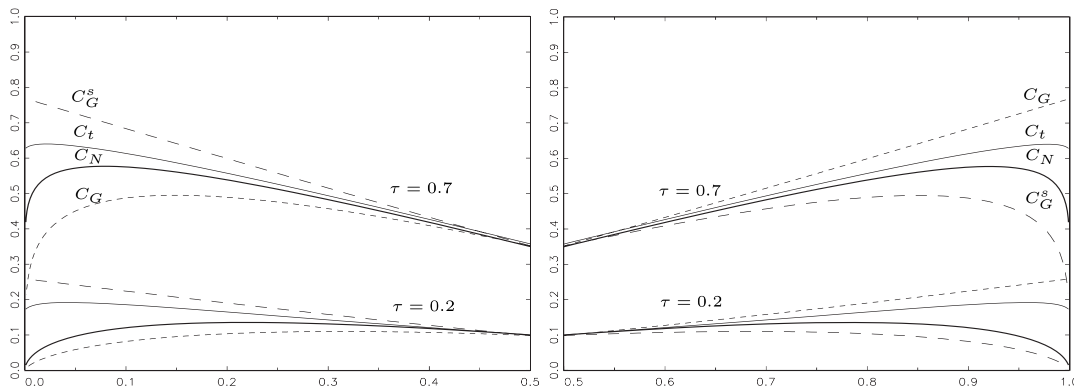

Figure 2. The differences: $\lambda_L(u) - u$ and $\lambda_U(u) - (1 - u)$ implied by C_N, C_t, C_G, and C_G^s.

no tail-dependence for any u. In Figure 2, we show the implied differences: $\lambda_L(u) - u$ and $\lambda_U(u) - (1 - u)$ of $C_o = C_N$, $C_o = C_G$, $C_o = C_G^s$, and $C_o = C_t$ with $\tau = 0.2, 0.7$. This figure indicates that these differences are all positive if $u \neq 0$ and $u \neq 1$. In other words, they imply both lower and upper tail-dependence except for the extreme cases: $u = 0, 1$. For these two extreme cases, we have $\lambda_L^* := \lim_{u \to 0^+} \lambda_L(u)$ and $\lambda_U^* := \lim_{u \to 1^-} \lambda_U(u)$ that measure the lower extreme-values dependence and the upper extreme-values dependence, respectively; see, e.g., Joe (1997). It is known that $C_o = C_N$ implies $\lambda_L^* = \lambda_U^* = 0$. In comparison, $C_o = C_G$ has $\lambda_L^* = 0$ and $0 \leq \lambda_U^* = 2 - 2^{\vartheta} < 1$, $C_o = C_G^s$ has $0 \leq \lambda_L^* = 2 - 2^{\vartheta_s} < 1$ and $\lambda_U^* = 0$, and $C_o = C_t$ has $\lambda_L^* = \lambda_U^* = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right)$; see, e.g., Embrechts, Lindskog, and McNeil (2003) and Schmidt (2004). Some studies refer to a copula with $\lambda_L^* = 0$ ($\lambda_U^* = 0$) as a "lower-tail-independent" (an "upper-tail-independent") copula. However, this terminology ignores the difference between the tail events and the extreme events. To avoid the resulting ambiguity, we will distinguish the tail-dependence from the extreme-values dependence, and measure the former by $\lambda_L(u)$ and $\lambda_U(u)$ and the latter by λ_L^* and λ_U^* in this study.

This discussion demonstrates that C_N, C_G, C_G^s, and C_t are all capable of interpreting concordance, but they may have quite different tail-dependence, extreme-value dependence, or both structures. In the literature, there are many other parametric copulae. For example, the Frank copula has a U-shaped dependence but no extreme-values dependence. The Clayton copula has an L-shaped dependence and lower-extreme-values dependence. The Clayton-survival copula has a J-shaped dependence and upper-extreme-values dependence. We may even accommodate various cross-dependence structures by using a mixed copula that combines different copulae with different weights; see, e.g., Hu (2006). The M test can be applied to any of these parametric copulae. Nevertheless, it seems unlikely to consider all possible copulae in empirical studies. It should be more reasonable to conduct certain sensible representative ones, and then check whether these copulae need to be re-specified. The M tests with properly selected ϕ's are useful for this task.

Following Jondeau and Rockinger (2006) and Patton (2006a), we can easily extend our discussions to dynamic copulae by specifying the copula parameters as certain dynamic functions of $\boldsymbol{x}_t$. Specifically, we can define the dynamic normal copula $C_N(u_1, u_2|\boldsymbol{x}_t; \theta)$ by using the DCC coefficient $\rho_t = \rho_t(\boldsymbol{x}_t; \theta)$, such as that of Tse and Tsui (2002):

$$\rho_t = (1 - \kappa_1 - \kappa_2)\kappa_o + \kappa_1 \rho_{t-1} + \kappa_2 \frac{\sum_{k=1}^{m} \varepsilon_{1,t-k}\varepsilon_{2,t-k}}{\sqrt{\left(\sum_{k=1}^{m} \varepsilon_{1,t-k}^2\right)\left(\sum_{k=1}^{m} \varepsilon_{2,t-k}^2\right)}}, \tag{21}$$

where $-1 \leq \kappa_o \leq 1$, $0 \leq \kappa_1 \leq 1$, $0 \leq \kappa_2 \leq 1$, $\kappa_1 + \kappa_2 \leq 1$, and $m = 2$, in place of the CCC coefficient ρ of $C_N(u_1, u_2; \rho)$. By using the dynamic parameters $\vartheta_t = 1 - \frac{2}{\pi}\arcsin(\rho_t)$ and $\vartheta_{s,t} = 1 - \frac{2}{\pi}\arcsin(\rho_t)$ in place of the parameters ϑ and ϑ_s of the static C_G and C_G^s, we can also define the dynamic Gumbel copula $C_G(u_1, u_2|\boldsymbol{x}_t; \theta)$ and the dynamic Gumbel-survival copula $C_G^s(u_1, u_2|\boldsymbol{x}_t; \theta)$, respectively. Similarly, we can define the dynamic t copula $C_t(u_1, u_2|\boldsymbol{x}_t; \theta)$ by using the same ρ_t to replace the CCC coefficient ρ of $C_t(u_1, u_2; \rho, \nu)$. By fixing the same ρ_t, these dynamic copulae have the same dynamic Kendall's tau $\tau(\boldsymbol{x}_t; \theta)$ in the population, but still have different tail-dependence structures characterized by the dynamic lower-u tail-dependence measures $\lambda_L(u|\boldsymbol{x}_t; \theta) = \frac{1}{u}C(u, u|\boldsymbol{x}_t; \theta)$ and the dynamic upper-u tail-dependence measures $\lambda_U(u|\boldsymbol{x}_t; \theta) = \frac{1}{1-u}C^s(1-u, 1-u|\boldsymbol{x}_t; \theta)$ for various C's. These dynamic copulae degenerate to the static counterparts as $\rho_t = \rho$ for all the t's. The tests introduced below are applicable to both the static and dynamic copulae.

3.2 Concordance Test and Tail-Dependence Tests

By definition, it is easy to see that the condition $\mathbb{E}[\phi_{ot}|\boldsymbol{I}^{t-1}] = 0$ is satisfied for all the following ϕ's:

$$\phi_\tau(\boldsymbol{u}_t|\boldsymbol{x}_t; \theta) = 4C(\boldsymbol{u}_t|\boldsymbol{x}_t; \theta) - 1 - \tau(\boldsymbol{x}_t; \theta), \tag{22}$$

$$\phi_{L(u)}(\boldsymbol{u}_t|\boldsymbol{x}_t; \theta) = \frac{1}{u}I(u_{1t} < u)I(u_{2t} < u) - \lambda_L(u|\boldsymbol{x}_t; \theta), \quad u \in (0, 0.5], \tag{23}$$

and

$$\phi_{U(u)}(\boldsymbol{u}_t|\boldsymbol{x}_t; \theta) = \frac{1}{1-u}[I(u_{1t} \geq u)I(u_{2t} \geq u)] - \lambda_U(u|\boldsymbol{x}_t; \theta), \quad u \in [0.5, 1), \tag{24}$$

where $\boldsymbol{u}_t = (u_{1t}, u_{2t})^\top$, under the null hypothesis. In what follows, we refer to the M test with $\phi = \phi_\tau$, $\phi = \phi_{L(u)}$, and $\phi = \phi_{U(u)}$ as the M_τ test, the $M_L(u)$ test, and the $M_U(u)$ test, respectively.

For the M_τ test, we have the derivatives: $\nabla_{\theta^\top}\phi_t = 4\nabla_{\theta^\top}C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta) - \nabla_{\theta^\top}\tau(\boldsymbol{x}_t;\theta)$ and $p_{it} = 4\frac{\partial}{\partial u_{it}}C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)$. Accordingly, we can estimate the expectations $\mathbb{E}[\nabla_{\theta^\top}\phi_t]$, $\mathbb{E}[p_{it}f_{it}w_{it}^\top]$, $\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top]$, and $\mathbb{E}[p_{it}\mathcal{F}_{it}]$ and hence the estimation-uncertainty-associated parameters: η_{co} and η_{io} by using their sample counterparts. Given these estimators, the transformation $\hat{\varphi}_t$ in (14) and hence the M_τ test statistic are immediately computable.

For the $M_{L(u)}$ test with some $u \in (0, 0.5]$, we have $\nabla_{\theta^\top}\phi_t = -\frac{1}{u}\nabla_{\theta^\top}C(u,u|\boldsymbol{x}_t;\theta)$ and $p_{it} = -\frac{1}{u}\delta(u_{it}-u)I(u_{jt} < u)$, where $i,j = 1,2$ and $j \neq i$. Recall that $\delta(\cdot)$ is the Dirac delta function. Denote $d_{it}(u_i,u) = \int_0^u c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}u_j$, where $c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)$ is the copula density function of $C(\boldsymbol{u}|\boldsymbol{x}_t;\theta)$ and $\boldsymbol{u} = (u_1,u_2)^\top$. By definition, we have

$$\mathbb{E}[p_{it}f_{it}|\boldsymbol{I}^{t-1}] = -\frac{1}{u}\iint_{[0,1]^2}\delta(u_i-u)I(u_j<u)f_{\varepsilon_i}(F_{\varepsilon_i}^{-1}(u_i|\boldsymbol{x}_t;\beta_i)|\boldsymbol{x}_t;\beta_i)c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}\boldsymbol{u},$$

under the null hypothesis (as $\lambda = \lambda_o$). Because $d_{it}(u_i,u) = \int_0^1 I(u_j<u)c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}u_j$, we can re-express this conditional expectation as

$$\mathbb{E}[p_{it}f_{it}|\boldsymbol{I}^{t-1}] = -\frac{1}{u}\int_0^1 \delta(u_i-u)\left\{d_{it}(u_i,u)f_{\varepsilon_i}(F_{\varepsilon_i}^{-1}(u_i|\boldsymbol{x}_t;\beta_i)|\boldsymbol{x}_t;\beta_i)\right\}\mathrm{d}u_i. \tag{25}$$

Denote $q_{it}^u := F_{\varepsilon_i}^{-1}(u|\boldsymbol{x}_t;\beta_i)$, $f_{it}^u := f_{it}|_{\varepsilon_{it}=q_{it}^u}$, $\mathcal{F}_{it}^u := \mathcal{F}_{it}|_{\varepsilon_{it}=q_{it}^u}$, and $d_{it}^u := d_{it}(u,u)$. By using (25), we can set $\epsilon = u_i$, $\epsilon_o = u$, and $\mu(\epsilon) = d_{it}(u_i,u)f_{\varepsilon_i}(F_{\varepsilon_i}^{-1}(u_i|\boldsymbol{x}_t;\beta_i)|\boldsymbol{x}_t;\beta_i)$ and utilize the sifting property (8) to show that $\mathbb{E}[p_{it}f_{it}|\boldsymbol{I}^{t-1}] = -\frac{1}{u}d_{it}^u f_{it}^u$. Accordingly, we have $\mathbb{E}[p_{it}f_{it}w_{it}^\top] = -\frac{1}{u}\mathbb{E}[d_{it}^u f_{it}^u w_{it}^\top]$ under the null hypothesis. Following the same argument, it is easy to see that $\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top] = -\frac{1}{u}\mathbb{E}[d_{it}^u f_{it}^u q_{it}^u z_{it}^\top]$ and $\mathbb{E}[p_{it}\mathcal{F}_{it}] = -\frac{1}{u}\mathbb{E}[d_{it}^u\mathcal{F}_{it}^u]$ under the null hypothesis. Recall that $\mathbb{E}[p_{it}f_{it}w_{it}^\top] = \mathbb{E}[p_{it}f_{it}]\mathbb{E}[w_{it}^\top]$ and $\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top] = \mathbb{E}[p_{it}f_{it}\varepsilon_{it}]\mathbb{E}[z_{it}^\top]$ if ε_{it} is independent of $\boldsymbol{x}_t$. In this case, $\mathbb{E}[p_{it}f_{it}w_{it}^\top] = -\frac{1}{u}\mathbb{E}[d_{it}^u f_{it}^u]\mathbb{E}[w_{it}^\top]$ and $\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top] = -\frac{1}{u}\mathbb{E}[d_{it}^u f_{it}^u q_{it}^u]\mathbb{E}[z_{it}^\top]$. Importantly, these expectations are free of the Dirac delta function, and hence can be directly estimated using their sample counterparts in practical applications. Given the estimators of $\mathbb{E}[\nabla_{\theta^\top}\phi_t]$ and these expectations, the $M_{L(u)}$ test becomes immediately applicable.

For the $M_{U(u)}$ test with some $u \in (0.5, 1]$, we have $\nabla_{\theta^\top}\phi_t = -\frac{1}{1-u}\nabla_{\theta^\top}C(u,u|\boldsymbol{x}_t;\theta)$ and $p_{it} = \frac{1}{1-u}\delta(u_{it}-u)I(u_{it} \geq u)$, $i,j = 1,2$ ($j \neq i$). Denote $e_{it}(u_i,u) = \int_u^1 c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}u_j$ and $e_{it}^u := e_{it}(u,u)$. Similar to the $M_{L(u)}$ test, we can apply the sifting property to show that $\mathbb{E}[p_{it}f_{it}w_{it}^\top] = \frac{1}{1-u}\mathbb{E}[e_{it}^u f_{it}^u w_{it}^\top]$, $\mathbb{E}[p_{it}f_{it}\varepsilon_{it}z_{it}^\top] = \frac{1}{1-u}\mathbb{E}[e_{it}^u f_{it}^u q_{it}^u z_{it}^\top]$, and $\mathbb{E}[p_{it}\mathcal{F}_{it}] = \frac{1}{1-u}\mathbb{E}[e_{it}^u\mathcal{F}_{it}^u]$ under the null hypothesis. Given the estimators of $\mathbb{E}[\nabla_{\theta^\top}\phi_t]$ and these expectations, the $M_{U(u)}$ test is applicable.

The M_τ, $M_{L(u)}$, and $M_{U(u)}$ tests are designed to check the criteria of concordance, lower-u tail-dependence, and upper-u tail-dependence individually. These individual tests are expected to be powerful against the copula mis-specifications in the directions of concordance, lower-u tail-dependence, and upper-u tail-dependence, respectively. Therefore, they are useful to identify the possible causes of copula mis-specification and to refine the mis-specified copula. In financial applications, the M_τ test may also be used to evaluate the adequacy of copula in characterizing the market co-movements in normal times. There are some studies that check the VaR validation by using a graphical comparison between the observed tail frequencies and the theoretical tail probabilities, implied by the copula used to evaluate VaR, at different confidence levels $u \in (0,1)$; see, e.g., Cherubini and Luciano (2001). Clearly, the $M_L(u)$ and $M_U(u)$ tests formalize such a graphical comparison method, and the index u may be interpreted as the associated confidence level. As such, these tail-dependence tests should be particularly useful for evaluating the VaR validation and the related risk-management applications.

This demonstrates some potential importance and applicability of these individual tests. Nonetheless, it is quite possible that a mis-specified copula may satisfy the criterion of concordance but fall short of certain tail-dependence criteria, as implied by the discussion of Section 3.1 and as we will show in the simulation. Similarly, it is also possible that a mis-specified copula may satisfy the tail-dependence criterion for certain u's but fail to satisfy this criterion for other u's. Therefore, in addition to the individual tests, it is also important to evaluate various criteria simultaneously. In our approach, we can easily establish such a test by basing the M test on certain multi-dimensional testing indicators. In particular, we may base the M test on the following $2p$-dimensional ϕ:

$$\phi_{LU} := (\phi_{L(v_1)}, \ldots, \phi_{L(v_p)}, \phi_{U(1-v_p)}, \ldots, \phi_{U(1-v_1)})^\top, \tag{26}$$

for some $v_i \in (0, 0.5)$, $v_i < v_{i+1}$, and $i = 1, 2, \ldots, p$. We refer to this test as the M_{LU} test. The M_{LU} test statistic can be easily computed by re-defining the transformation $\hat{\varphi}_t$ in (14) as a $2p \times 1$ vector composed of the $\hat{\varphi}_t$'s implied by the $M_{L(v_1)}, \ldots, M_{L(v_p)}, M_{U(1-v_p)}, \ldots$, and $M_{U(1-v_1)}$ tests. This test statistic has the asymptotic null distribution $\chi^2(2p)$. Unlike the above-mentioned M tests, the M_{LU} test can check the copula mis-specifications for various tail-dependence structures at the same time.

In addition, we can also easily extend the M_τ, $M_{L(u)}$, $M_{U(u)}$, and M_{LU} tests to the multivariate copula tests by replacing the bivariate Kendall's tau and tail-dependence measures with their multivariate generalizations. Specifically, as noted by Nelsen (2002), the bivariate Kendall's tau in (19) has a multivariate generalization:

$$\tau = \frac{1}{2^{n-1} - 1} \left[2^n \int_0^1 \cdots \int_0^1 C(\boldsymbol{u}) \mathrm{d}C(\boldsymbol{u}) - 1 \right], \tag{27}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n)^\top$ for some $n \geq 2$. Accordingly, we can base the multivariate concordance test on the testing indicator:

$$\phi_\tau(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta) = \frac{1}{2^{n-1}-1}\left[2^n C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta) - 1\right] - \tau(\boldsymbol{x}_t;\theta), \tag{28}$$

where $\boldsymbol{u}_t = (u_{1t}, u_{2t}, \ldots, u_{nt})^\top$ and $\tau(\boldsymbol{x}_t;\theta)$ is defined by (27) using $C(\boldsymbol{u}|\boldsymbol{x}_t;\theta)$ in place of $C(\boldsymbol{u})$. By replacing the bivariate copula with the n-dimensional copula, we can also define the multivariate tail-dependence measures $\lambda_L(u|\boldsymbol{x}_t;\theta) = \frac{1}{u}C(u,u,\ldots,u|\boldsymbol{x}_t;\theta)$ and $\lambda_U(u|\boldsymbol{x}_t;\theta) = \frac{1}{1-u}C^s(1-u, 1-u, \ldots, 1-u|\boldsymbol{x}_t;\theta)$ and base the multivariate tail-dependence tests on the extended $\phi_{L(u)}$ and $\phi_{U(u)}$:

$$\phi_{L(u)}(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta) = \frac{1}{u}\prod_{i=1}^{n} I(u_{it} < u) - \lambda_L(u|\boldsymbol{x}_t;\theta), \quad u \in (0, 0.5],$$

and

$$\phi_{U(u)}(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta) = \frac{1}{1-u}\prod_{i=1}^{n} I(u_{it} \geq u) - \lambda_U(u|\boldsymbol{x}_t;\theta), \quad u \in [0.5, 1).$$

These multivariate copula tests degenerate to the bivariate copula tests when $n = 2$.

For the multivariate tail-dependence tests, the expectations $\mathbb{E}[p_{it} f_{it} w_{it}^\top]$, $\mathbb{E}[p_{it} f_{it} \varepsilon_{it} z_{it}^\top]$, and $\mathbb{E}[p_{it}\mathcal{F}_{it}]$ are of the same forms as those in the bivariate case but the transformations d_{it}^u and e_{it}^u must be generalized as $d_{it}^u = \int_0^u \cdots \int_0^u c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}\boldsymbol{u}_{-i}$ and $e_{it}^u = \int_u^1 \cdots \int_u^1 c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}\boldsymbol{u}_{-i}$ evaluated at $u_i = u$, in which $\boldsymbol{u}_{-i}$ is the $(n-1)\times 1$ sub-vector of $\boldsymbol{u}$ precluding the element u_i. Moreover, $C^s(1-u, 1-u, \ldots, 1-u|\boldsymbol{x}_t;\theta) = \int_u^1 \cdots \int_u^1 c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}\boldsymbol{u}$; see also Cherubini, Luciano, and Vecchiato (2004, p.142) for more discussions regarding the multivariate survival copula.

As pointed out by a referee, it may also be worth noting that, by using the standardized residuals vector $\boldsymbol{\varepsilon}_t$ to replace the PITs vector $\boldsymbol{u}_t$ in the testing indicator ϕ_t and by applying the generalized first-order asymptotics to re-derive the asymptotic distribution of the resulting "$\sqrt{T}\hat{D}_T$", we might also establish a class of moment-based tests for the entire multivariate conditional density model that is composed of the marginal models and the copula model. Similar to the concordance and tail-dependence tests, we may also base this class of tests on the correlation coefficients of $\boldsymbol{\varepsilon}_t$ or the conditional quantile-exceedances of $\boldsymbol{\varepsilon}_t$ to check the entire multivariate conditional density model in various directions. To focus on testing the copula model, we will not further pursue the details of this testing approach in the rest of this paper.

Table 1. Empirical sizes and powers of the M_τ test.

	H_o : $C_o = C_I$			H_o : $C_o = C_N$			C_I(uncorrected)			C_N(uncorrected)		
C_o	T =500	1000	2500	T =500	1000	2500	T =500	1000	2500	T =500	1000	2500
C_I	**8.1**	**7.4**	**6.6**	**7.0**	**6.8**	**7.9**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**
C_N (τ_1)	43.8	68.7	95.8	**6.8**	**8.5**	**7.6**	0.2	2.9	31.4	**0.0**	**0.0**	**0.0**
C_N (τ_2)	100.0	100.0	100.0	**7.6**	**8.4**	**7.5**	100.0	100.0	100.0	**0.0**	**0.0**	**0.0**
C_G (τ_1)	43.4	70.8	96.6	7.5	6.7	8.3	0.1	2.6	27.5	0.0	0.0	0.0
C_G (τ_2)	100.0	100.0	100.0	7.3	9.3	7.2	100.0	100.0	100.0	0.0	0.0	0.0
C_t (τ_1)	38.2	61.2	94.6	6.4	8.1	5.8	0.2	2.8	28.6	0.0	0.0	0.0
C_t (τ_2)	100.0	100.0	100.0	7.8	9.2	9.4	100.0	100.0	100.0	0.0	0.0	0.0

Notes: The bold entries represent the empirical sizes in percentages, and the others are the empirical powers in percentages. The "uncorrected" blocks correspond to the tests without the correction for estimation uncertainty.

4 MONTE CARLO SIMULATION

In this simulation, we assess the finite sample performance of the proposed method. The CMD model being tested is of the form:

$$F_{\boldsymbol{y}}(\boldsymbol{y}|\boldsymbol{x}_t;\lambda) = C\left(F_{y_1}(y_1|\boldsymbol{x}_t;\gamma_1), F_{y_2}(y_2|\boldsymbol{x}_t;\gamma_2);\theta\right),$$

in which the marginal models $F_{y_i}(y_i|\boldsymbol{x}_t;\gamma_i)$ are specified to have the AR(1) conditional mean $m_{it} = \alpha_{mo}+\alpha_{m1}y_{i,t-1}$, the GARCH(1,1) conditional variance $h_{it} = \alpha_{ho}+\alpha_{h1}h_{i,t-1}+\alpha_{h2}(y_{i,t-1} - m_{i,t-1})^2$, and the i.i.d. $N(0,1)$ standardized error ε_{it} for both $i = 1,2$. The copula model being tested includes $C = C_I$ and the static C_N, and the true copula includes $C_o = C_I$, C_N, C_G, and C_t with the static parameter $\rho = 0.1$ and 0.5 (or equivalently the static Kendall's tau $\tau_1 := 0.0638$ and $\tau_2 := 0.3317$), where C_t has the degrees of freedom $\nu = 4$. The marginal models are set to be correctly specified in the sense of assumption [A] with $(\alpha_{mo}, \alpha_{m1}, \alpha_{ho}, \alpha_{h1}, \alpha_{h2}) = (0.01, 0.05, 0.05, 0.85, 0.1)$. This experiment design essentially follows Chen, Fan, and Patton (2004) for further discussion and comparison.

Given $T = 500, 1000, 2500$, the 5% nominal level, and one thousand replications, we present the empirical sizes and powers of the M_τ test in Table 1 and show the simulation results of the M_{LU} test with $(v_1, v_2, v_3, 1-v_3, 1-v_2, 1-v_1) = (0.1, 0.3, 0.5, 0.5, 0.7, 0.9)$ and the associated $M_{L(u)}$ and $M_{U(u)}$ tests in Table 2. To demonstrate the importance of correcting the effect of estimation uncertainty, we also report the empirical sizes and powers of the uncorrected tests in these two tables. Specifically, these uncorrected tests standardize the statistic $\sqrt{T}\hat{D}_T = T^{-1/2}\sum_{t=1}^{T}\hat{\phi}_t$ using the sample counterpart of $\mathbb{E}[\phi_{ot}\phi_{ot}^\top]$, rather than the estimator of the estimation-uncertainty-corrected asymptotic variance Ω_o.

Table 2. Empirical sizes and powers of the M_{LU}, $M_{L(u)}$, and $M_{U(u)}$ tests.

		$H_o : C_o = C_I$			$H_o : C_o = C_N$			C_I(uncorrected)			C_N(uncorrected)		
C_o	test	T =500	1000	2500	T =500	1000	2500	T =500	1000	2500	T =500	1000	2500
C_I	M_{LU}	**10.9**	**6.6**	**5.6**	**8.2**	**8.1**	**6.5**	**8.9**	**4.9**	**3.1**	**7.6**	**4.1**	**2.1**
	$M_L(0.1)$	**10.1**	**6.2**	**6.0**	**7.4**	**6.0**	**5.5**	**9.5**	**5.8**	**3.0**	**6.6**	**4.8**	**3.0**
	$M_L(0.3)$	**5.5**	**4.7**	**5.4**	**5.8**	**4.3**	**5.9**	**1.9**	**2.2**	**2.3**	**0.7**	**0.8**	**1.2**
	$M_L(0.5)$	**6.5**	**5.7**	**5.7**	**5.6**	**5.2**	**4.9**	**1.7**	**0.7**	**1.6**	**0.7**	**0.4**	**0.4**
	$M_U(0.5)$	**5.2**	**5.9**	**4.6**	**6.5**	**5.2**	**5.6**	**1.7**	**1.3**	**0.9**	**0.4**	**0.3**	**0.3**
	$M_U(0.7)$	**5.5**	**6.6**	**5.8**	**4.2**	**4.2**	**5.2**	**2.1**	**2.5**	**2.0**	**1.0**	**0.4**	**1.3**
	$M_U(0.9)$	**11.6**	**6.3**	**5.0**	**7.6**	**6.6**	**4.8**	**11.1**	**5.7**	**2.5**	**7.4**	**5.7**	**3.4**
C_N (τ_1)	M_{LU}	19.1	38.1	86.0	**7.3**	**7.1**	**5.6**	11.9	26.9	77.6	**5.2**	**3.7**	**1.5**
	$M_L(0.1)$	8.4	12.1	34.3	**5.4**	**4.5**	**5.1**	5.0	7.1	27.8	**4.3**	**3.2**	**2.9**
	$M_L(0.3)$	20.5	34.2	70.6	**5.0**	**5.0**	**5.0**	9.1	17.9	53.6	**0.4**	**0.7**	**0.6**
	$M_L(0.5)$	21.2	36.7	69.1	**5.2**	**6.1**	**5.7**	6.5	16.5	44.2	**0.8**	**0.2**	**0.3**
	$M_U(0.5)$	17.6	32.1	68.1	**5.1**	**6.2**	**5.8**	5.0	12.8	43.7	**0.5**	**0.7**	**0.3**
	$M_U(0.7)$	18.8	33.7	69.9	**5.5**	**6.0**	**5.1**	7.3	19.3	53.9	**0.9**	**0.7**	**0.2**
	$M_U(0.9)$	9.5	13.1	33.6	**5.7**	**6.6**	**5.2**	6.5	8.9	26.1	**4.8**	**4.1**	**2.5**
C_N (τ_2)	M_{LU}	100.0	100.0	100.0	**6.2**	**6.3**	**4.9**	100.0	100.0	100.0	**2.5**	**1.7**	**1.2**
	$M_L(0.1)$	96.8	99.9	100.0	**6.9**	**5.3**	**3.8**	92.0	99.9	100.0	**2.4**	**1.8**	**1.0**
	$M_L(0.3)$	100.0	100.0	100.0	**5.8**	**5.6**	**4.9**	100.0	100.0	100.0	**0.6**	**0.5**	**0.6**
	$M_L(0.5)$	100.0	100.0	100.0	**5.3**	**5.6**	**5.8**	100.0	100.0	100.0	**0.4**	**0.2**	**0.3**
	$M_U(0.5)$	100.0	100.0	100.0	**5.0**	**6.0**	**5.4**	99.9	100.0	100.0	**0.3**	**0.3**	**0.0**
	$M_U(0.7)$	100.0	100.0	100.0	**4.0**	**5.9**	**6.5**	100.0	100.0	100.0	**0.0**	**0.3**	**0.7**
	$M_U(0.9)$	96.1	100.0	100.0	**5.2**	**5.5**	**5.0**	92.8	99.9	100.0	**2.1**	**1.9**	**0.8**
C_G (τ_1)	M_{LU}	24.6	52.6	97.1	8.4	12.8	25.9	14.0	35.2	92.1	5.6	6.1	11.5
	$M_L(0.1)$	7.1	7.1	18.8	9.8	8.6	10.8	5.2	3.7	14.2	9.1	7.1	8.0
	$M_L(0.3)$	12.7	23.8	52.0	6.5	9.5	10.6	4.4	10.8	36.1	1.1	2.6	2.1
	$M_L(0.5)$	18.6	33.4	65.2	4.7	5.4	5.5	7.6	14.9	40.7	0.2	0.5	0.2
	$M_U(0.5)$	18.3	31.8	65.1	5.3	6.3	6.0	7.0	13.3	38.9	0.6	0.3	0.3
	$M_U(0.7)$	29.9	52.0	88.3	5.9	7.8	9.2	11.9	31.7	75.9	0.8	0.8	1.3
	$M_U(0.9)$	24.5	48.5	92.8	9.5	16.5	42.9	13.9	38.3	89.5	3.4	5.9	25.7
C_G (τ_2)	M_{LU}	100.0	100.0	100.0	45.4	79.3	99.8	100.0	100.0	100.0	15.9	35.7	88.7
	$M_L(0.1)$	77.5	97.5	100.0	19.9	34.5	70.6	66.5	95.5	100.0	13.7	24.0	55.1
	$M_L(0.3)$	100.0	100.0	100.0	15.9	23.2	47.7	99.5	100.0	100.0	2.3	5.0	16.0
	$M_L(0.5)$	100.0	100.0	100.0	4.6	5.9	5.7	99.8	100.0	100.0	0.2	0.1	0.1
	$M_U(0.5)$	100.0	100.0	100.0	5.7	5.9	6.2	100.0	100.0	100.0	0.1	0.4	0.8
	$M_U(0.7)$	100.0	100.0	100.0	18.4	31.1	65.2	100.0	100.0	100.0	1.6	4.9	20.7
	$M_U(0.9)$	100.0	100.0	100.0	50.4	85.0	99.5	100.0	100.0	100.0	17.8	55.9	97.5
C_t (τ_1)	M_{LU}	43.7	88.7	100.0	22.8	61.5	99.1	24.1	74.9	99.9	4.2	13.4	69.5
	$M_L(0.1)$	40.3	72.7	99.2	17.9	38.0	84.8	27.2	64.3	98.6	5.4	21.7	71.5
	$M_L(0.3)$	31.4	54.3	93.0	8.6	8.3	14.9	13.4	03.5	84.2	0.8	1.2	2.6
	$M_L(0.5)$	23.1	35.0	70.7	7.1	6.1	5.8	8.5	15.3	44.3	0.6	0.3	0.5
	$M_U(0.5)$	18.2	31.4	67.0	5.5	6.2	5.8	6.4	12.3	41.9	0.5	0.3	0.4
	$M_U(0.7)$	31.2	53.5	92.6	7.8	10.0	16.1	14.0	32.3	82.1	1.3	1.3	2.3
	$M_U(0.9)$	41.7	71.6	98.9	18.7	41.2	84.6	28.9	62.7	98.2	8.4	23.3	71.6
C_t (τ_2)	M_{LU}	100.0	100.0	100.0	19.2	37.3	89.2	100.0	100.0	100.0	4.1	4.7	24.7
	$M_L(0.1)$	99.4	100.0	100.0	14.1	24.5	63.6	09.1	100.0	100.0	2.6	7.7	35.2
	$M_L(0.3)$	100.0	100.0	100.0	7.2	8.9	14.2	99.9	100.0	100.0	0.8	0.5	1.7
	$M_L(0.5)$	100.0	100.0	100.0	6.4	5.0	7.6	99.9	100.0	100.0	0.3	0.5	0.6
	$M_U(0.5)$	100.0	100.0	100.0	5.0	5.9	5.5	99.6	100.0	100.0	0.1	0.0	0.2
	$M_U(0.7)$	100.0	100.0	100.0	7.3	8.5	15.3	100.0	100.0	100.0	0.5	0.9	2.3
	$M_U(0.9)$	99.7	100.0	100.0	14.2	27.0	61.8	99.3	100.0	100.0	2.7	7.1	35.0

Notes: The bold entries represent the empirical sizes in percentages, and the others are the empirical powers in percentages. The "uncorrected" blocks correspond to the tests without the correction for estimation uncertainty.

From these two tables, it can be seen that these uncorrected tests are substantially under-sized in most cases. Importantly, this distortion may not be remedied, but could be instead damaged, by the increase in T. On the contrary, the empirical sizes of the M test are quite close to the 5% nominal level for both the continuously differentiable testing indicator: ϕ_τ and the discrete testing indicators: $\phi_{L(u)}$ and $\phi_{U(u)}$ and for both the one-dimensional testing indicators: ϕ_τ, $\phi_{L(u)}$, and $\phi_{U(u)}$ and the multi-dimensional testing indicator: ϕ_{LU}. A mild exception appears in the case where the M_{LU}, $M_L(0.1)$, and $M_U(0.9)$ tests have the empirical sizes: 10.9%, 10.1%, and 11.6% when $C_o = C_I$ and $T = 500$. Nonetheless, this distortion disappears as $T = 1000$. This size performance does not only demonstrate the importance of correcting the estimation uncertainty effect but also support the validity of the generalized first-order asymptotics used in Section 2.

These two tables also show that the empirical powers of the uncorrected tests are generally smaller than those of the corrected tests. In addition, the M tests have quite good performance against the copula mis-specifications that they are designed to detect. The empirical powers of the M_τ test against the mis-specified C_I rapidly increase with the Kendall's tau of C_o and the sample size, regardless of whether $C_o = C_N$, C_G, or C_t. This means that this test can successfully capture the concordance structures ignored by C_I. Interestingly, the empirical "powers" of the M_τ test against the mis-specified C_N are close to (or slightly greater than) the 5% level in both cases where $C_o = C_G$ and $C_o = C_t$. This is quite a reasonable result because it reflects the fact that C_N is also capable of interpreting the concordance structure implied by C_G and C_t. Nevertheless, it also reminds us that we should not completely rely on the concordance test to conclude the adequacy of copula, as discussed in Section 3.2.

Indeed, what C_N cannot interpret are the tail-dependence structures of C_G and C_t. As expected, the M_{LU} test is quite powerful in these directions. Table 2 shows that, given $T = 2500$, the M_{LU} test is of the empirical power: 99.8% against the mis-specified C_N when $C_o = C_G$ and $\tau = \tau_2$, and it has the empirical power: 99.1% against the mis-specified C_N when $C_o = C_t$ and $\tau = \tau_1$. This shows that the M_{LU} test can successfully discriminate between the copulae with different tail dependence structures for proper T's and τ's.

Table 2 also shows that, given $\tau = \tau_2$ and $T = 2500$, the $M_L(0.1)$, $M_L(0.3)$, $M_L(0.5)$, $M_U(0.5)$, $M_U(0.7)$, and $M_U(0.9)$ tests are, respectively, of the empirical powers: 70.6%, 47.7%, 5.7%, 6.2%, 65.2%, and 99.5% against the mis-specified C_N when $C_o = C_G$. Quite remarkably, this "J-shaped" power performance is consistent with the dissimilarity between the J-shaped dependence of $C_o = C_G$ and the U-shaped dependence of C_N, as discussed in Section 3.1; see also Figure 2 for why the $M_U(0.9)$ test has a higher power than the $M_L(0,1)$ test. Given $\tau = \tau_1$ and $T = 2500$, these tests are, respectively, of the empirical powers: 84.8%, 14.9%, 5.8%, 5.8%, 16.1%, and 84.6% against the mis-specified C_N when

$C_o = C_t$. This "U-shaped" performance is also consistent with the symmetric lower- and upper-extreme-values dependence of $C_o = C_t$ that cannot be interpreted by C_N. Such a power performance is quite encouraging. It shows the usefulness of the proposed individual tests in shedding light on the possible directions of copula mis-specification. This property is important because C_o is unknown in practical applications and we have to identify the possible causes of mis-specification before re-specifying the mis-specified copula model.

Theoretically, we may not completely and fairly compare our tests with the density-estimates-based tests of Chen, Fan, and Patton (2004) on the same basis because they are based on differently designed contexts. Unlike our tests, their tests are designed for the semi-parametric CMD context that replaces the parametric standardized errors distribution $F_{\varepsilon_i}(\cdot|\boldsymbol{x}_t;\beta_i)$ with the empirical distribution function of the standardized residuals $\hat{\varepsilon}_{it}$'s. Because the parametric and semi-parametric modelling approaches both exist in the copula studies, these two classes of tests may have different potential applicants and should not be exclusive to each other. Nonetheless, we may still discuss the performance of the tests that is to some extent based on the same simulation design.

Chen, Fan, and Patton (2004) introduced two density-estimates-based tests: "Test 1" and "Test 2". The former is a multivariate-density-estimate-based consistent test, while the latter is a univariate-density-estimate-based inconsistent test. By basing on some particular choices of kernel and bandwidth, their simulation shows that given $T = 500, 2500$, and 5000, Test 1 (Test 2) has, respectively, the empirical sizes: 0.0%, 0.4%, and 5.2% (0.4%, 2.0%, and 3.2%) under the null hypothesis $C_o = C_N$ when $\rho = 0.1$, and the empirical sizes become 0.0%, 2.4%, and 4.0% (1.6%, 2.4%, and 2.4%) as $\rho = 0.5$; see Tables 5 and 6 of their paper. This indicates that although the density-estimates-based tests are asymptotically free of the standardized error distribution specifications, they could be obviously undersized in small and moderate samples. In comparison, although the moment-based tests have much better size performance, they require the correctly specified standardized error distributions. Such trade-offs are common between the parametric and semi-parametric statistical methods. From these aspects, we interpret the moment-based tests as complements, rather than substitutes, to the density-estimates-based tests.

5 AN EMPIRICAL APPLICATION

In this section, we apply the concordance and tail-dependence tests to an empirical study of stock market relationships. Our discussions will mainly focus on the bivariate C_N, C_G, C_G^s, and C_t, and finally extend to the trivariate C_N and C_t. Similar to Hu (2006), we view C_N, C_G, and C_G^s as the representative ones with the U-shaped, J-shaped, and L-shaped dependence, respectively. Recall that the t copula also has the U-shaped dependence. If the true copula has the L-shaped (J-shaped) dependence, then the cross-dependence of downside markets is stronger (weaker) than that of the upside markets. By contrast, if the true copula is of the U-shaped dependence, then there will be no such asymmetry. This asymmetry (symmetry) is conceptually very close to the correlation asymmetry (symmetry), studied by Longin and Solnik (2001) and Ang and Chen (2002), which is known to have important implications for portfolio diversification and risk management.

The data used in our analysis include seven major stock price indices: the Standard & Poor 500 (SP) and Russell 2000 (RS) of the U.S., the Financial Times Stock Exchange 100 (FT) of the U.K., the Compagnie des Agents de Change 40 (CA) of France, the Nikkei 225 (NK) of Japan, the Hang Seng (HS) of Hong Kong, and the Taiwan weighted (TW) from January 1, 1995 through December 31, 2003. These data are obtained from Yahoo!Finance. Let P_{it} be the closing price of stock index i at date t and in the local currency. This empirical study is based on the daily returns:

$$y_{it} = 100 \times (\ln P_{it} - \ln P_{i,t-1}),$$

where t denotes the t-th common calendar trading date of these markets in the sample. The sample size is $T = 1915$. We consider twenty-one pairs of returns $\boldsymbol{y}_t = (y_{1t}, y_{2t})$, including the U.S. returns: SP-RS, the U.S.-European returns: SP-FT, SP-CA, RS-FT, RS-CA, the European returns: FT-CA, the Asian returns: NK-HS, NK-TW, and HS-TW, the U.S.-Asian returns: SP-NK, SP-HS, SP-TW, RS-NK, RS-HS, and RS-TW, and the European-Asian returns: FT-NK, FT-HS, FT-TW, CA-NK, CA-HS, and CA-TW.

In Table 3, we show the first four sample moments of returns. Not surprisingly, these returns are all leptokurtically distributed. This evidence precludes the marginal normality and hence the bivariate normality for all the return combinations. To see if the return series are i.i.d., we further check the null of serial independence using the Ljung-Box (1978) test, the McLeod-Li (1982) test, and the time reversibility (TR) test of Chen (2003). These tests are, respectively, powerful against serial correlation, volatility clustering, and time irreversibility (asymmetry in dynamic dependence, such as the leverage effect). These power directions will provide us with useful information to establish suitable marginal models when the null of serial independence is rejected. We show the test statistics in the same table, and describe the tests in the footnotes to this table.

Given the 5% significance level, these tests indicate that most return series are likely to be serially correlated, volatility-clustered, and time irreversible. The case of NK is the only case that has volatility clustering but serial uncorrelatedness and time reversibility. Nevertheless, all of these returns are dynamically dependent and need to be explained using some

Table 3. Descriptive statistics and serial independence test statistics.

	SP	RS	FT	CA	NK	HS	TW
mean	0.045	0.042	0.020	0.033	-0.033	0.024	-0.010
std. dev	1.269	1.330	1.268	1.631	1.606	1.961	1.855
skewness	-0.068	-0.704	0.020	-0.007	0.057	0.296	-0.130
kurtosis	6.176	9.613	6.218	6.967	5.374	13.414	6.408
Q_{11}	20.146*	23.309*	31.099*	22.851*	9.459	24.812*	22.371*
Q_{22}	245.264*	178.847*	758.547*	352.473*	104.040*	463.286*	88.020*
TR	37.130*	43.886*	18.548*	15.517*	9.502	27.590*	31.313*

Notes: Let $\hat{r}_1(k)$ be the lag-k sample autocorrelation of the return series $\{y_t\}$, and denote $T_k := T-k$. The Ljung-Box test statistic is $Q_{11} := T(T+2)\sum_{k=1}^{m}\hat{r}_1^2(k)/T_k$, and the McLeod-Li test statistic Q_{22} replaces $\{y_t\}$ in Q_{11} with $\{y_t^2\}$. These two test statistics are evaluated at $m = 10$. Under the null of serial independence, the TR test statistic TR := $\sum_{k=1}^{m} T_k\hat{\Theta}_k'(\hat{\Sigma} - 2\hat{\Gamma})^{-1}\hat{\Theta}_k \xrightarrow{d} \chi^2(m)$, where $\hat{\Theta}_k := T_k^{-1}\sum_{t=k+1}^{T}\Phi(y_t, y_{t-k})$, $\hat{\Sigma} := T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\Phi(y_t, y_s)^2$, $\hat{\Gamma} := T^{-1}\sum_{t=1}^{T}\left[T^{-1}\sum_{s=1}^{T}\Phi(y_t, y_s)\right]^2$, and $\Phi(y_t, y_{t-k}) := \tilde{\beta}(y_t - y_{t-k})/[1 + \tilde{\beta}^2(y_t - y_{t-k})^2]$, has the asymptotic null distribution $\chi^2(m)$. This statistic is evaluated at $\tilde{\beta} = 0.5$ and $m = 5$; see Chen (2003) for the finite sample performance. The symbol * represents significance at the 5% level. The 95% critical values of $\chi^2(5)$ and $\chi^2(10)$ are, respectively, 11.0705 and 18.3070.

suitable GARCH-type models. We consider the AR-GARCH and AR-EGARCH models with different orders, and check their adequacy using the standardized-residuals-based counterparts of the Ljung-Box, McLeod-Li, and TR tests, referred to as the diagnostic tests, that are corrected for estimation uncertainty. In Table 4, we show the Gaussian QMLEs and the diagnostic test statistics of the selected GARCH-type models: SP**1**, RS**1**, FT**1**, CA**1**, NK**1**, HS**1**, and TW**1**. We observe that NK**1** is the only case that has the GARCH specification and the other cases all have the EGARCH specification. The diagnostic tests accept that these GARCH-type models can successfully interpret the serial dependence of $y_{it}|I_i^{t-1}$ for all the return series.

However, as discussed in Section 2, the marginal models for the copula analysis must be based on the same information set $\boldsymbol{I}^{t-1}$. Therefore, we have to further check whether the "$y_{it}|I_i^{t-1}$ model" is mis-specified for $y_{it}|\boldsymbol{I}^{t-1}$ due to ignoring certain important I_j^{t-1}-based variables, $i \neq j$ and $i, j = 1, 2$, before the bivariate copula analysis. This detection is essential but often ignored in the empirical studies; Patton (2006a) is an important exception that emphasizes the role of this detection in applying the conditional Sklar theorem. In our study, we conduct the causality-in-mean and causality-in-variance tests of Cheung and Ng (1996) for this detection. In Table 5, we show the causality test statistics for the twenty-one pairs of models generated by different combinations of SP**1**, RS**1**, FT**1**, CA**1**, NK**1**, HS**1**, and TW**1**.

Table 4. The QMLEs of the marginal models.

	SP**1**	RS**1**	FT**1**	CA**1**	NK**1**	HS**1**	TW**1**	RS**2**	FT**2**	CA**2**	NK**2**	HS**2**
α_{mo}	·	0.063	·	·	·	·	·	0.066	·	·	·	·
	·	(0.022)	·	·	·	·	·	(0.022)	·	·	·	·
$y_{i,t-1}$	·	0.115	·	·	·	0.040	·	0.129	−0.148	−0.120	·	−0.071
	·	(0.024)	·	·	·	(0.026)	·	(0.025)	(0.026)	(0.032)	·	(0.025)
$y_{i,t-2}$	·	·	·	·	·	·	0.069	·	−0.032	·	·	·
	·	·	·	·	·	·	(0.026)	·	(0.022)	·	·	·
$y_{j,t-1}$	·	·	·	·	·	·	·	−0.035	0.304	0.348	0.410	0.475
	·	·	·	·	·	·	·	(0.022)	(0.024)	(0.045)	(0.030)	(0.033)
$y_{j,t-2}$	·	·	·	·	·	·	·	·	·	·	·	0.061
	·	·	·	·	·	·	·	·	·	·	·	(0.031)
$y_{j,t-3}$	·	·	·	·	·	·	·	·	·	−0.073	·	0.098
	·	·	·	·	·	·	·	·	·	(0.034)	·	(0.030)
α_{ho}	−0.064	−0.113	−0.079	−0.075	0.145	−0.091	−0.033	−0.109	−0.083	−0.074	0.126	−0.078
	(0.018)	(0.031)	(0.014)	(0.018)	(0.052)	(0.026)	(0.020)	(0.030)	(0.014)	(0.014)	(0.049)	(0.022)
$\ln h_{i,t-i}$	0.958	0.977	0.983	0.979	·	0.973	0.957	0.979	0.981	0.979	·	0.986
	(0.007)	(0.010)	(0.004)	(0.006)	·	(0.009)	(0.012)	(0.009)	(0.004)	(0.006)	·	(0.005)
$h_{i,t-i}$	·	·	·	·	0.873	·	·	·	·	·	0.878	·
	·	·	·	·	(0.035)	·	·	·	·	·	(0.038)	·
$e^2_{i,t-i}$	·	·	·	·	0.072	·	·	·	·	·	0.068	·
	·	·	·	·	(0.020)	·	·	·	·	·	(0.020)	·
$\varepsilon_{i,t-1}$	−0.167	−0.096	−0.112	−0.089	·	−0.097	−0.084	−0.094	−0.116	−0.081	·	−0.088
	(0.024)	(0.022)	(0.019)	(0.022)	·	(0.062)	(0.020)	(0.020)	(0.019)	(0.024)	·	(0.049)
$\lvert\varepsilon_{i,t-1}\rvert$	0.104	0.157	0.107	0.120	·	0.124	0.109	0.150	0.110	0.118	·	0.135
	(0.025)	(0.042)	(0.018)	(0.027)	·	(0.100)	(0.029)	(0.039)	(0.019)	(0.021)	·	(0.045)
$\varepsilon_{i,t-2}$	·	·	·	·	·	0.013	·	·	·	·	·	0.039
	·	·	·	·	·	(0.057)	·	·	·	·	·	(0.077)
$\lvert\varepsilon_{i,t-2}\rvert$	·	·	·	·	·	0.037	·	·	·	·	·	−0.015
	·	·	·	·	·	(0.087)	·	·	·	·	·	(0.072)
$y^2_{j,t-1}$	·	·	·	·	·	·	·	·	·	·	·	−0.045
	·	·	·	·	·	·	·	·	·	·	·	(0.011)
$y^2_{j,t-2}$	·	·	·	·	·	·	·	·	·	·	·	0.098
	·	·	·	·	·	·	·	·	·	·	·	(0.030)
Q^*_{11}	11.644	6.269	14.143	12.451	3.679	16.721	14.780	6.065	17.350	14.041	12.676	18.266
Q^*_{22}	9.031	11.233	8.304	8.710	14.640	9.387	10.675	10.468	7.618	7.294	8.397	13.478
TR*	2.055	10.125	4.866	7.201	9.829	7.865	9.555	10.385	3.122	5.539	3.230	3.194
Kiefer-Salmon	115.859	1424.243	103.775	289.372	352.394	593.794	843.573	1284.299	203.936	892.647	518.037	582.132
skewness	-0.165	-0.795	0.063	0.081	-0.081	-0.134	-0.134	-0.775	0.135	0.216	-0.058	0.015
kurtosis	3.960	6.893	4.081	4.859	5.069	5.718	6.224	6.678	4.513	6.275	5.515	5.730

Notes: The entries (in the parentheses) are the first-stage Gaussian QMLEs (and their standard deviations) of the GARCH-type models: (1) $m_{it} = 0$, $h_{it} = h^e_{it} := \exp(\alpha_{ho} + \alpha_{h1} \ln h_{i,t-1} + \alpha_{h2}\varepsilon_{i,t-1} + \alpha_{h3}|\varepsilon_{i,t-1}|)$ for SP**1**, CA**1**, and FT**1**, (2) $m_{it} = \alpha_{mo} + \alpha_{m1} y_{i,t-1}$, $h_{it} = h^e_{it}$ for RS**1**, (3) $m_{it} = \alpha_{m1} y_{i,t-1}$, $h_{it} = h^{e-\mathrm{HS1}}_{it} := \exp(\alpha_{ho} + \alpha_{h1} \ln h_{i,t-1} + \alpha_{h2}\varepsilon_{i,t-1} + \alpha_{h3}|\varepsilon_{i,t-1}| + \alpha_{h4}\varepsilon_{i,t-2} + \alpha_{h5}|\varepsilon_{i,t-2}|)$ for HS**1**, (4) $m_{it} = \alpha_{m1} y_{i,t-2}$, $h_{it} = h^e_{it}$ for TW**1**, and (5) $m_{it} = 0$, $h_{it} = h^g_{it} := \alpha_{ho} + \alpha_{h1} h_{i,t-1} + \alpha_{h2} e^2_{i,t-1}$, where $e_{it} := y_{it} - m_{it}$, for NK**1**, (6) $m_{it} = \alpha_{mo} + \alpha_{m1} y_{i,t-1} + \alpha_{m2} y_{j,t-1}$, $h_{it} = h^e_{it}$, where y_{jt} denotes the HS return, for RS**2**, (7) $m_{it} = \alpha_{m1} y_{i,t-1} + \alpha_{m2} y_{i,t-2} + \alpha_{m3} y_{j,t-1}$, $h_{it} = h^e_{it}$, where y_{jt} denotes the SP return, for FT**2**, (8) $m_{it} = \alpha_{m1} y_{i,t-1} + \alpha_{m2} y_{j,t-1} + \alpha_{m3} y_{j,t-3}$, $h_{it} = h^e_{it}$, where y_{jt} denotes the SP return, for CA**2**, (9) $m_{it} = \alpha_{m1} y_{j,t-1}$, $h_{it} = h^g_{it}$, where y_{jt} denotes the SP return, for NK**2**, (10) $m_{it} = \alpha_{m1} y_{i,t-1} + \alpha_{m2} y_{j,t-1} + \alpha_{m3} y_{j,t-2} + \alpha_{m4} y_{j,t-3}$, $h_{it} = h^{e-\mathrm{HS2}}_{it} := \exp(\alpha_{ho} + \alpha_{h1} \ln h_{i,t-1} + \alpha_{h2}\varepsilon_{i,t-1} + \alpha_{h3}|\varepsilon_{i,t-1}| + \alpha_{h4}\varepsilon_{i,t-2} + \alpha_{h5}|\varepsilon_{i,t-2}| + \alpha_{h6} y^2_{j,t-1} + \alpha_{h7} y^2_{j,t-2})$, where y_{jt} denotes the SP return, for HS**2**; see the Appendix for the estimation method. The test statistics Q^*_{11}, Q^*_{22}, and TR* are, respectively, the standardized-residuals-based diagnostic test statistics corresponding to the original-returns-based test statistics: Q_{11}, Q_{22}, TR shown in Table 3 with the correction for estimation uncertainty; see Chen (2003) for the the TR test and Chen (2005) for the Q^*_{11} and Q^*_{22} tests. Under the null of conditional normality, the Kiefer-Salmon test statistic is of the form: $\frac{T}{6}(\hat{\mu}_3 - 3\hat{\mu}_1)^2 + \frac{T}{24}(\hat{\mu}_4 - 6\hat{\mu}_2 + 3)^2 \xrightarrow{d} \chi^2(2)$, where $\hat{\mu}_i := T^{-1}\sum_{t=1}^{T} \hat{\varepsilon}^i_t$ and $\hat{\varepsilon}_t$ denotes the standardized residuals; as discussed by Bontemps and Meddahi (2005), this test is applicable to the standardized residuals of the GARCH-type models. "skewness" and "kurtosis" are, respectively, the sample skewness and kurtosis coefficients of the standardized residuals.

Table 4. The QMLEs of the marginal models (continued).

	TW**2**	FT**3**	CA**3**	NK**3**	HS**3**	TW**3**	NK**4**	HS**4**	TW**4**	NK**5**	HS**5**	TW**5**
α_{mo}	·	·	·	·	·	·	·	·	·	·	·	·
	·	·	·	·	·	·	·	·	·	·	·	·
$y_{i,t-1}$	0.016	−0.115	−0.099	·	−0.063	0.010	·	−0.023	·	·	−0.032	·
	(0.026)	(0.026)	(0.026)	·	(0.027)	(0.024)	·	(0.027)	·	·	(0.027)	·
$y_{i,t-2}$	0.047	−0.059	·	·	·	0.060	·	·	0.068	·	·	0.070
	(0.027)	(0.023)	·	·	·	(0.025)	·	·	(0.025)	·	·	(0.025)
$y_{j,t-1}$	0.236	0.251	0.297	0.363	0.393	0.227	0.263	0.269	0.184	0.207	0.176	0.160
	(0.037)	(0.023)	(0.030)	(0.032)	(0.031)	(0.034)	(0.033)	(0.031)	(0.035)	(0.025)	(0.025)	(0.027)
$y_{j,t-2}$	0.028	−0.050	−0.050	·	0.035	0.056	·	·	·	·	0.069	·
	(0.037)	(0.020)	(0.027)	·	(0.030)	(0.036)	·	·	·	·	(0.027)	·
$y_{j,t-3}$	0.052	·	·	·	0.094	·	·	·	·	·	0.046	·
	(0.033)	·	·	·	(0.029)	·	·	·	·	·	(0.021)	·
$y_{j,t-5}$	0.034	·	·	·	·	·	·	·	·	·	·	·
	(0.035)	·	·	·	·	·	·	·	·	·	·	·
$y_{j,t-6}$	·	·	·	·	·	·	·	·	·	·	−0.051	·
	·	·	·	·	·	·	·	·	·	·	(0.021)	·
$y_{j,t-8}$	·	·	·	·	·	·	·	·	·	·	−0.046	·
	·	·	·	·	·	·	·	·	·	·	(0.022)	·
α_{ho}	−0.021	−0.085	−0.077	0.131	−0.086	−0.026	0.138	−0.086	−0.037	0.136	−0.089	−0.035
	(0.010)	(0.014)	(0.017)	(0.050)	(0.028)	(0.027)	(0.053)	(0.025)	(0.020)	(0.054)	(0.025)	(0.018)
$\ln h_{i,t-i}$	0.990	0.982	0.977	·	0.980	0.926	·	0.972	0.958	·	0.975	0.965
	(0.005)	(0.004)	(0.007)	·	(0.007)	(0.019)	·	(0.009)	(0.012)	·	(0.008)	(0.011)
$h_{i,t-i}$	·	·	·	0.876	·	·	0.878	·	·	0.880	·	·
	·	·	·	(0.037)	·	·	(0.036)	·	·	(0.037)	·	·
$e^2_{i,t-i}$	·	·	·	0.069	·	·	0.067	·	·	0.066	·	·
	·	·	·	(0.019)	·	·	(0.019)	·	·	(0.019)	·	·
$\varepsilon_{i,t-1}$	−0.125	−0.105	−0.084	·	−0.092	−0.102	·	−0.064	−0.081	·	−0.082	−0.075
	(0.047)	(0.019)	(0.021)	·	(0.056)	(0.024)	·	(0.053)	(0.020)	·	(0.054)	(0.018)
$\|\varepsilon_{i,t-1}\|$	0.075	0.113	0.124	·	0.188	0.128	·	0.179	0.112	·	0.158	0.099
	(0.064)	(0.018)	(0.025)	·	(0.051)	(0.036)	·	(0.049)	(0.031)	·	(0.050)	(0.029)
$\varepsilon_{i,t-2}$	0.020	·	·	·	0.029	·	·	−0.020	·	·	0.002	·
	(0.045)	·	·	·	(0.093)	·	·	(0.084)	·	·	(0.086)	·
$\|\varepsilon_{i,t-2}\|$	0.164	·	·	·	−0.052	·	·	−0.026	·	·	−0.005	·
	(0.076)	·	·	·	(0.085)	·	·	(0.081)	·	·	(0.077)	·
$\varepsilon_{i,t-3}$	0.061	·	·	·	·	·	·	·	·	·	·	·
	(0.096)	·	·	·	·	·	·	·	·	·	·	·
$\|\varepsilon_{i,t-3}\|$	−0.197	·	·	·	·	·	·	·	·	·	·	·
	(0.069)	·	·	·	·	·	·	·	·	·	·	·
$y^2_{j,t-1}$	·	·	·	·	0.022	·	·	·	·	·	−0.045	·
	·	·	·	·	(0.007)	·	·	·	·	·	(0.011)	·
$y^2_{j,t-2}$	·	·	·	·	−0.021	0.006	·	·	·	·	0.098	·
	·	·	·	·	(0.007)	(0.003)	·	·	·	·	(0.030)	·
Q^*_{11}	13.849	13.834	13.505	12.029	17.707	13.351	10.812	17.610	9.630	13.402	16.696	8.667
Q^*_{22}	12.089	6.481	5.959	12.867	12.638	12.607	10.632	11.518	12.635	11.093	10.563	15.067
TR*	9.266	3.980	7.266	5.366	5.469	8.237	8.232	4.969	9.991	5.833	5.555	9.149
Kiefer-Salmon	898.081	157.237	413.149	506.394	603.496	1113.570	311.382	583.099	940.772	349.734	510.009	944.371
skewness	-0.232	0.147	0.150	-0.076	-0.067	-0.197	0.028	-0.120	-0.153	-0.030	-0.150	-0.219
kurtosis	6.240	4.333	5.209	5.488	5.774	6.712	4.933	5.700	6.404	5.060	5.534	6.394

Notes: The entries (in the parentheses) are the Gaussian QMLEs (and their standard deviations) of the GARCH-type models: (11) $m_{it} = \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{i,t-2} + \alpha_{m3}y_{j,t-1} + \alpha_{m4}y_{j,t-2} + \alpha_{m5}y_{j,t-3} + \alpha_{m6}y_{j,t-5}$, $h_{it} = \exp(\alpha_{ho} + \alpha_{h1}\ln h_{i,t-1} + \alpha_{h2}\varepsilon_{i,t-1} + \alpha_{h3}|\varepsilon_{i,t-1}| + \alpha_{h4}\varepsilon_{i,t-2} + \alpha_{h5}|\varepsilon_{i,t-2}| + \alpha_{h6}\varepsilon_{i,t-3} + \alpha_{h7}|\varepsilon_{i,t-3}|)$, where y_{jt} is the SP return, for TW**2**, (12) $m_{it} = \alpha_{mo} + \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{i,t-2} + \alpha_{m3}y_{j,t-1} + \alpha_{m4}y_{j,t-2}$, $h_{it} = h^e_{it}$, where y_{jt} is the RS return, for FT**3**, (13) $m_{it} = \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{j,t-1} + \alpha_{m3}y_{j,t-2}$, $h_{it} = h^e_{it}$, where y_{jt} is the RS return, for CA**3**, (14) $m_{it} = \alpha_{m1}y_{j,t-1}$, $h_{it} = h^g_{it}$, where y_{jt} is the RS return, for NK**3**, (15) $m_{it} = \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{j,t-1} + \alpha_{m3}y_{j,t-2} + \alpha_{m4}y_{j,t-3}$, $h_{it} = h^{e-\mathrm{HS2}}_{it}$, where y_{jt} is the RS return, for HS**3**, (16) $m_{it} = \alpha_{mo} + \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{i,t-2} + \alpha_{m3}y_{j,t-1} + \alpha_{m4}y_{j,t-2}$, $h_{it} = \exp(\alpha_{ho} + \alpha_{h1}\ln h_{i,t-1} + \alpha_{h2}\varepsilon_{i,t-1} + \alpha_{h3}|\varepsilon_{i,t-1}| + \alpha_{h4}y^2_{j,t-2})$, where y_{jt} is the RS return, for TW**3**, (17) $m_{it} = \alpha_{m1}y_{j,t-1}$, $h_{it} = h^g_{it}$, where y_{jt} is the FT return, for NK**4**, (18) $m_{it} = \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{j,t-2}$, $h_{it} = h^{e-\mathrm{HS1}}_{it}$, where y_{jt} is the FT return, for HS**4**, (19) $m_{it} = \alpha_{m1}y_{i,t-2} + \alpha_{m2}y_{j,t-1}$, $h_{it} = h^e_{it}$, where y_{jt} is the FT return, for TW**4**, (20) $m_{it} = \alpha_{m1}y_{j,t-1}$, $h_{it} = h^g_{it}$, where y_{jt} is the CA return, for NK**5**, (21) $m_{it} = \alpha_{m1}y_{i,t-1} + \alpha_{m2}y_{j,t-1} + \alpha_{m3}y_{j,t-2} + \alpha_{m4}y_{j,t-3} + \alpha_{m5}y_{j,t-6} + \alpha_{m6}y_{j,t-8}$, $h_{it} = h^{e-\mathrm{HS2}}_{it}$, where y_{jt} denotes the CA return, for HS**5**, and (19) $m_{it} = \alpha_{m1}y_{i,t-2} + \alpha_{m2}y_{j,t-1}$, $h_{it} = h^e_{it}$, where y_{jt} is the CA return, for TW**5**.

Table 5. The causality test statistics for the $y_{it}|I_i^{t-1}$ models.

	causality-in-mean							causality-in-variance						
	SP**1**	RS**1**	FT**1**	CA**1**	NK**1**	HS**1**	TW**1**	SP**1**	RS**1**	FT**1**	CA**1**	NK**1**	HS**1**	TW**1**
SP**1**	$\cdot$	4.283	143.836*	135.126*	201.675*	244.488*	83.478*	$\cdot$	3.975	11.429	14.001	26.795*	76.401*	7.322
RS**1**	12.778	$\cdot$	120.898*	112.366*	155.011*	166.851*	71.262*	12.044	$\cdot$	6.014	10.776	21.012*	42.062*	10.816
FT**1**	8.642	5.204	$\cdot$	14.961	79.006*	94.594*	37.747*	8.995	7.451	$\cdot$	3.466	19.487*	5.977	2.377
CA**1**	9.885	4.485	9.542	$\cdot$	85.079*	82.756*	52.697*	6.193	2.416	8.029	$\cdot$	15.185	7.857	15.600
NK**1**	11.866	13.641	16.602	7.395	$\cdot$	10.932	14.722	4.581	3.468	8.621	2.443	$\cdot$	15.623	8.312
HS**1**	16.341	19.533*	14.376	18.842*	10.610	$\cdot$	19.449*	5.070	9.443	11.265	5.539	5.805	$\cdot$	20.847*
TW**1**	8.461	19.337*	9.675	12.071	8.982	5.391	$\cdot$	10.741	6.122	6.922	7.836	7.119	23.119*	$\cdot$

Notes: Let $\tilde{\varepsilon}_{1t}$ and $\tilde{\varepsilon}_{2t}$ be, respectively, the standardized residuals of the $y_{1t}|I_1^{t-1}$ model and the $y_{2t}|I_2^{t-1}$ model. Denote $\hat{r}_{1,12}(k)$ as the lag-k sample cross-correlation coefficient of $\tilde{\varepsilon}_{1t}$ and $\tilde{\varepsilon}_{2,t-k}$, and $\hat{r}_{2,12}(k)$ as the lag-k sample cross-correlation coefficient of $\tilde{\varepsilon}_{1t}^2$ and $\tilde{\varepsilon}_{2,t-k}^2$. The causality-in-mean and causality-in-variance test statistics are, respectively, of the forms: $T\sum_{k=1}^m \hat{r}_{1,12}^2(k)$ and $T\sum_{k=1}^m \hat{r}_{2,12}^2(k)$. These two test statistics have the asymptotic null distribution $\chi^2(m)$ under the null hypothesis that y_{2t} does not cause y_{1t}; see Cheung and Ng (1996). In this study, we set $m = 10$. For the upper-triangle (lower-triangle) parts of the test statistics matrices, the row denotes the $y_{1t}|I_1^{t-1}$ $\left(y_{2t}|I_2^{t-1}\right)$ model, and the column denotes the $y_{2t}|I_2^{t-1}$ $\left(y_{1t}|I_1^{t-1}\right)$ model. The symbol * represents significance at the 5% level. The 95% critical value of $\chi^2(10)$ is 18.3070.

This table shows that the causality test statistics are insignificant at the 5% level for only the following four combinations: SP**1**-RS**1**, FT**1**-CA**1**, NK**1**-HS**1**, and NK**1**-TW**1**. By contrast, the causality-in-mean test statistics are obviously significant for the remaining seventeen combinations. The test result indicates that the conditional means of the Asian (European) returns are likely to be strongly influenced by the past U.S. and European (U.S.) return shocks. It can also be seen from this table that the conditional variance of NK**1** (HS**1**) is significantly influenced by the past volatility shocks of SP**1**, RS**1**, and FT**1** (SP**1** and RS**1**); moreover, the conditional variance of HS**1** is also affected by the past volatility shocks of TW**1**, and vice versa. As such, we have to modify these seventeen pairs of models by taking into account this empirical evidence.

For this purpose, we extend the conditional mean (variance) specifications of the models that have the significant causality-in-mean (causality-in-variance) test statistics by considering the associated $y_{j,t-k}$'s ($y_{j,t-k}^2$'s), for some $k \leq 10$, as additional explanatory variables. Then, we re-check the extended "$y_{it}|\boldsymbol{I}^{t-1}$ model" using the causality tests and the diagnostic tests. This method successfully generates sixteen pair of models: SP**1**-FT**2**, SP**1**-CA**2**, SP**1**-NK**2**, SP**1**-HS**2**, SP**1**-TW**2**, RS**1**-FT**3**, RS**1**-CA**3**, RS**1**-NK**3**, RS**2**-HS**3**, RS**1**-TW**3**, FT**1**-NK**4**, FT**1**-HS**4**, FT**1**-TW**4**, CA**1**-NK**5**, CA**1**-HS**5**, and CA**1**-TW**5**. The Gaussian QMLEs and the diagnostic test statistics of these models are shown in Table 4, and the causality test statistics are reported in Table 6. These test statistics are all insignificant at the 5% level. The returns combination: HS-TW is the only exception in that we are unable to find the suitable $y_{it}|\boldsymbol{I}^{t-1}$ models by this method, so that it is precluded in the rest of this empirical study. The following discussion will be based on SP**1**-RS**1**, FT**1**-CA**1**, NK**1**-HS**1**, NK**1**-TW**1**, and the sixteen above-mentioned combinations.

Table 6. The causality test statistics for the $y_{it}|\boldsymbol{I}^{t-1}$ models.

	causality-in-mean							causality-in-variance						
	SP	RS	FT	CA	NK	HS	TW	SP	RS	FT	CA	NK	HS	TW
SP	·	·	9.957	14.966	6.963	18.027	15.360	·	·	7.739	16.339	4.724	5.669	7.219
RS	·	·	8.957	17.424	8.985	18.015	17.428	·	·	6.553	9.771	4.972	17.300	7.274
FT	9.160	4.959	·	·	5.278	15.520	5.231	6.881	8.507	·	·	17.799	6.977	2.521
CA	11.367	5.382	·	·	7.420	15.564	11.635	5.447	2.709	·	·	15.008	6.513	12.334
NK	9.594	10.238	14.229	5.936	·	·	·	4.730	3.402	11.699	3.163	·	·	·
HS	13.570	18.042	12.740	17.024	·	·	·	7.565	6.898	10.127	4.383	·	·	·
TW	7.850	17.551	8.961	12.596	·	·	·	9.844	6.172	6.973	8.010	·	·	·

Notes: The combinations SP-FT, SP-CA, SP-NK, SP-HS, SP-TW, RS-FT, RS-CA, RS-NK, RS-HS, RS-TW, FT-NK, FT-HS, FT-TW, CA-NK, CA-HS, and CA-TW are, respectively, corresponding to the following models: SP**1**-FT**2**, SP**1**-CA**2**, SP**1**-NK**2**, SP**1**-HS**2**, SP**1**-TW**2**, RS**1**-FT**3**, RS**1**-CA**3**, RS**1**-NK**3**, RS**2**-HS**3**, RS**1**-TW**3**, FT**1**-NK**4**, FT**1**-HS**4**, FT**1**-TW**4**, CA**1**-NK**5**, CA**1**-HS**5**, and CA**1**-TW**5**. For the upper-triangle (lower-triangle) parts of the test statistics matrices, the row denotes the $y_{1t}|\boldsymbol{I}^{t-1}$ $(y_{2t}|\boldsymbol{I}^{t-1})$ model, and the column denotes the $y_{2t}|\boldsymbol{I}^{t-1}$ $(y_{1t}|\boldsymbol{I}^{t-1})$ model.

Given these twenty pairs of GARCH-type models, we can accomplish the fully-specified marginal models by specifying suitable distributions for their standardized errors. For this purpose, we first check the normality of the standardized errors using the Kiefer-Salmon (1983) test. Table 4 shows that the Kiefer-Salmon test statistic is significant at any reasonable level for all the cases. This means that the standardized errors of the marginal models may still be leptokurtic, asymmetric, or both, and hence need to be explained by some non-normal distributions. Because of its flexibility in accommodating asymmetry and leptokurtosis, we consider the use of Hansen's (1994) unconditional skewed t distribution. The QMLEs for this standardized errors distribution are shown in Table 7. We also check this distribution specification by using Bai's (2003) test; see Table 7. The resulting test statistics are insignificant at the 5% level for all the cases considered. This leads us to conclude the adequacy of the skewed t distribution in this empirical study. Given the marginal models and the QMLEs shown in Table 4 and Table 7, we can further define the $\hat{\boldsymbol{u}}_t$'s, the conditional PIT estimates, for our copula analysis.

It is common to check the null of serial independence as the first step in the time series analysis. Similarly, we may also test the null of $C_o = C_I$ as the first step in the copula analysis. In Table 8, we show the concordance and tail-dependence test statistics for this null hypothesis. Note that the M_τ, $M_{L(u)}$, and $M_{U(u)}$ test statistics used in this section all take the form of M_T' and have the asymptotic null distribution $N(0,1)$; the M_{LU} test statistic has the asymptotic null distribution $\chi^2(6)$. This table shows that the M_τ test statistic are positive and significant at the 5% level in most cases except for SP**1**-NK**2**, SP**1**-TW**2**, RS**1**-NK**3**, RS**1**-TW**3**, and FT**1**-TW**4**. In comparison, the M_{LU} test is significant at any reasonable level for all twenty pairs of marginal models (combinations of returns). This reflects the fact that these stock markets are closely related. As such, it is worth investigating their cross-dependence structures using more sensible copula models.

Table 7. The QMLEs of the skewed t distribution for standardized errors.

	SP**1**	RS**1**	FT**1**	CA**1**	NK**1**	HS**1**	TW**1**	RS**2**	FT**2**	CA**2**	NK**2**	HS**2**
ς	−0.100	−0.127	−0.045	−0.018	0.018	0.018	0.024	−0.136	−0.002	0.024	0.037	0.014
	(0.016)	(0.027)	(0.019)	(0.022)	(0.028)	(0.029)	(0.033)	(0.026)	(0.025)	(0.029)	(0.032)	(0.028)
ν	8.759	6.271	10.186	9.125	7.127	5.567	5.144	6.505	8.869	7.932	6.861	5.595
	(0.162)	(0.199)	(0.205)	(0.193)	(0.193)	(0.143)	(0.147)	(0.202)	(0.219)	(0.200)	(0.203)	(0.141)
Bai's test	0.968	1.457	0.860	0.904	0.573	1.060	1.442	1.435	0.654	1.034	0.890	0.603
	TW**2**	FT**3**	CA**3**	NK**3**	HS**3**	TW**3**	NK**4**	HS**4**	TW**4**	NK**5**	HS**5**	TW**5**
ς	0.025	−0.021	0.021	0.045	0.028	0.016	0.040	0.003	0.016	0.023	−0.014	0.013
	(0.038)	(0.022)	(0.026)	(0.033)	(0.030)	(0.034)	(0.031)	(0.028)	(0.033)	(0.029)	(0.025)	(0.033)
ν	4.798	9.638	8.278	7.544	5.782	5.092	6.724	5.482	5.101	6.742	5.629	5.067
	(0.158)	(0.213)	(0.201)	(0.230)	(0.155)	(0.151)	(0.193)	(0.136)	(0.146)	(0.188)	(0.131)	(0.147)
Bai's test	1.336	0.686	0.871	0.772	0.840	1.347	0.576	0.742	1.159	0.613	0.704	1.241

Notes: The unconditional skewed t distribution for standardized errors is of the probability density function:

$$f_{\varepsilon_i}(\epsilon;\beta) = \begin{cases} \mathrm{bc}\left(1+\frac{1}{\nu-2}\left(\frac{\mathrm{b}\epsilon+\mathrm{a}}{1-\varsigma}\right)^2\right)^{-\frac{\nu+1}{2}}, & \epsilon < -\frac{\mathrm{a}}{\mathrm{b}}, \\ \mathrm{bc}\left(1+\frac{1}{\nu-2}\left(\frac{\mathrm{b}\epsilon+\mathrm{a}}{1+\varsigma}\right)^2\right)^{-\frac{\nu+1}{2}} & \epsilon \geq -\frac{\mathrm{a}}{\mathrm{b}}, \end{cases}$$

where $\mathrm{a} := 4\varsigma\mathrm{c}\left(\frac{\nu-2}{\nu-1}\right)$, $\mathrm{b} := \sqrt{1+3\varsigma^2-\mathrm{a}^2}$, and $\mathrm{c} := \Gamma\left(\frac{\nu+1}{2}\right)/\left(\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)\right)$; $\beta = (\varsigma,\nu)^\top$ with $\varsigma \in (-1,1)$ denoting the skewness parameter and $\nu \in (2,\infty)$ denoting the degrees of freedom. This distribution degenerates to the standardized t distribution with ν degrees of freedom as $\varsigma = 0$. The entries (in the parentheses) are the QMLEs (and their standard deviations) obtained by the second-stage QML estimation: $\hat{\beta}_{iT} := \mathrm{argmax}_{\beta_i \in B_i} L_{\beta,iT}(\beta_i|\hat{\alpha}_{iT})$; see the Appendix for the estimation method. "Bai's test" denotes the standardized-residuals-based distribution test statistic of Bai (2003, Theorem 4). The 95% critical value of this test statistic is 2.22.

Table 8. Tests for the null hypothesis: $C_o = C_I$.

	SP**1** RS**1**	SP**1** FT**2**	SP**1** CA**2**	SP**1** NK**2**	SP**1** HS**2**	SP**1** TW**2**	RS**1** FT**3**	RS**1** CA**3**	RS**1** NK**3**	RS**2** HS**3**
M_τ	13.156*	6.255*	6.608*	1.860	3.631*	1.735	6.936*	7.212*	1.227	3.027*
M_{LU}	712.368*	248.548*	256.257*	39.046*	53.392*	18.234*	225.699*	249.819*	43.704*	66.488*
$M_{L(0.1)}$	9.289*	5.760*	6.185*	2.693*	3.875*	2.398*	5.536*	5.697*	2.959*	4.131*
$M_{L(0.3)}$	14.548*	6.984*	6.830*	3.349*	3.104*	1.473	8.442*	8.226*	4.324*	3.568*
$M_{L(0.5)}$	13.974*	5.651*	5.407*	2.110*	2.229*	0.925	7.674*	7.949*	3.812*	4.226*
$M_{U(0.5)}$	14.214*	6.711*	6.962*	2.707*	3.226*	1.501	6.706*	6.871*	1.193	2.822*
$M_{U(0.7)}$	14.048*	7.622*	8.226*	1.506	3.461*	0.633	7.669*	8.725*	1.836	4.077*
$M_{U(0.9)}$	8.824*	6.889*	6.970*	2.540*	2.601*	2.515*	5.794*	5.935*	3.380*	2.829*
	RS**1** TW**3**	FT**1** CA**1**	FT**1** NK**4**	FT**1** HS**4**	FT**1** TW**4**	CA**1** NK**5**	CA**1** HS**5**	CA**1** TW**5**	NK**1** HS**1**	NK**1** TW**1**
M_τ	0.554	9.099*	3.102*	4.971*	1.696	3.292*	4.901*	2.370*	4.874*	2.400*
M_{LU}	20.855*	657.423*	95.190*	168.043*	28.528*	108.323*	145.587*	34.684*	214.092*	86.679*
$M_{L(0.1)}$	3.465*	8.603*	3.739*	4.886*	2.955*	4.539*	4.679*	2.829*	5.718*	3.776*
$M_{L(0.3)}$	2.151*	11.267*	5.590*	5.321*	1.145	5.188*	4.748*	2.405*	7.115*	3.954*
$M_{L(0.5)}$	2.132*	9.994*	3.664*	4.909*	1.988*	3.892*	4.528*	2.540*	5.622*	4.045*
$M_{U(0.5)}$	−0.514	10.969*	3.609*	5.604*	2.121*	4.038*	5.231*	2.117*	5.304*	3.027*
$M_{U(0.7)}$	0.447	12.157*	3.527*	5.868*	1.711	4.125*	5.422*	2.495*	6.244*	4.190*
$M_{U(0.9)}$	2.023*	9.632*	4.314*	5.059*	2.216*	4.665*	4.803*	2.190*	6.136*	3.126*

Notes: The symbol * represents significance at the 5% level. The 95% critical value of $\chi^2(6)$ is 12.5916.

Table 9. The test for the null hypothesis: CCC.

SP**1** RS**1**	SP**1** FT**2**	SP**1** CA**2**	SP**1** NK**2**	SP**1** HS**2**	SP**1** TW**2**	RS**1** FT**3**	RS**1** CA**3**	RS**1** NK**3**	RS**2** HS**3**
4.453*	1.942	1.965	2.194	5.942*	4.189*	4.310*	3.579	2.515	3.969*
RS**1** TW**3**	FT**1** CA**1**	FT**1** NK**4**	FT**1** HS**4**	FT**1** TW**4**	CA**1** NK**5**	CA**1** HS**5**	CA**1** TW**5**	NK**1** HS**1**	NK**1** TW**1**
2.034	3.754	2.188	6.046*	3.741	2.002	5.445*	2.782	2.040	3.041

Notes: The entries of this table are the realizations of the information matrix test statistic "IM_s" of Bera and Kim (2002, p.180). Under the null hypothesis of CCC, this test statistic has the asymptotic null distribution $\chi^2(1)$. The symbol * represents significance at the 5% level.

Before estimating the copula models, we first conduct the information matrix test of Bera and Kim (2002) to check whether the hypothesis of CCC holds for these twenty return combinations. The resulting test statistics are shown in Table 9. It can be seen from this table that this test significantly rejects this hypothesis for seven return combinations: SP**1**-RS**1**, SP**1**-HS**2**, SP**1**-TW**2**, RS**1**-FT**3**, RS**2**-HS**3**, FT**1**-HS**4**, and CA**1**-NK**5**, but not for the remaining return combinations, at the 5% significance level. According to this test result, we fit the dynamic C_N, C_G, C_G^s, and C_t to the former and estimate the static C_N, C_G, C_G^s, and C_t for the latter. In estimating the dynamic copulae, the DCC coefficient ρ_t is specified in the form of (21) because of its simplicity and good performance, as illustrated by Tse and Tsui (2002). The QMLEs of the static copulae and the dynamic copulae are, respectively, shown in Tables 10 and 11. We also show the M test statistics for the null hypotheses: $C_o = C_N$, $C_o = C_G$, $C_o = C_G^s$, and $C_o = C_t$ in Tables 12, 13, 14, and 15, respectively.

To avoid the difficulty in evaluating the derivative of the copula function $\nabla_{\theta^\top} C(\boldsymbol{u}|\boldsymbol{x}_t;\theta)$, we compute tail-dependence test statistics for the t copula (and the trivariate copulae) by using the representation $\nabla_{\theta^\top} C(\boldsymbol{u}|\boldsymbol{x}_t;\theta) = \mathbb{E}\left[\prod_{i=1}^n I(u_{it} < u_i)c(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)|\boldsymbol{I}^{t-1}\right]$. In addition, we use the expression $\mathbb{E}[\nabla_{\theta^\top}\phi_t] = -4\mathbb{E}[C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)l_{\theta t}^\top]$ in computing the M_τ test statistic for the t copula. To see this expression, note that the null hypothesis implies $\mathbb{E}[C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)|\boldsymbol{I}^{t-1}] = \iint_{[0,1]^2} C(\boldsymbol{u}|\boldsymbol{x}_t;\theta)c(\boldsymbol{u}|\boldsymbol{x}_t;\theta)\mathrm{d}\boldsymbol{u}$. By taking the derivative of this conditional expectation, we obtain $\nabla_{\theta^\top}\mathbb{E}[C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)|\boldsymbol{I}^{t-1}] = \mathbb{E}[\nabla_{\theta^\top}C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)|\boldsymbol{I}^{t-1}] + \mathbb{E}[C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)l_{\theta t}^\top|\boldsymbol{I}^{t-1}]$, where $l_{\theta t} := \nabla_\theta \ln c(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)$ denoted in ($A3$) of the Appendix. Moreover, by using the condition: $\mathbb{E}[\phi_\tau(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)|\boldsymbol{I}^{t-1}] = 0$ and its derivative taken with respect to θ, we have another result: $4\nabla_{\theta^\top}\mathbb{E}[C(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)|\boldsymbol{I}^{t-1}] = \nabla_{\theta^\top}\tau(\boldsymbol{x}_t;\theta)$. The expression is due to these two results.

The main test results are summarized as follows. First, the M_τ test statistic becomes insignificant at the 5% level for all the combinations, regardless of whether C_N, C_G, C_G^s, or C_t is being tested. This demonstrates that these bivariate copulae are all capable of

Table 10. The QMLEs of the static C_N, C_G, C_G^s, and C_t.

		SP1 FT2	SP1 CA2	SP1 NK2	RS1 CA3	RS1 NK3	RS1 TW3	FT1 CA1	FT1 NK4	FT1 TW4	CA1 NK5	CA1 TW5	NK1 HS1	NK1 TW1
C_N	ρ	0.479	0.486	0.143	0.461	0.171	0.095	0.740	0.273	0.111	0.278	0.143	0.430	0.254
		(0.015)	(0.016)	(0.019)	(0.017)	(0.020)	(0.019)	(0.007)	(0.018)	(0.019)	(0.017)	(0.019)	(0.017)	(0.017)
	$\hat{L}_{cT}$	0.128	0.132	0.010	0.116	0.014	0.004	0.392	0.038	0.006	0.040	0.010	0.102	0.033
C_G	θ	0.706	0.697	0.917	0.712	0.897	0.940	0.485	0.841	0.937	0.836	0.918	0.735	0.852
		(0.005)	(0.005)	(0.007)	(0.007)	(0.008)	(0.009)	(0.002)	(0.006)	(0.007)	(0.006)	(0.008)	(0.004)	(0.006)
	$\hat{L}_{cT}$	0.124	0.131	0.011	0.113	0.015	0.006	0.381	0.035	0.007	0.037	0.010	0.096	0.032
C_G^s	θ_s	0.706	0.694	0.911	0.715	0.903	0.939	0.472	0.833	0.930	0.830	0.911	0.734	0.861
		(0.012)	(0.013)	(0.015)	(0.013)	(0.016)	(0.017)	(0.006)	(0.013)	(0.015)	(0.014)	(0.016)	(0.012)	(0.015)
	$\hat{L}_{cT}$	0.118	0.126	0.013	0.109	0.014	0.008	0.402	0.040	0.008	0.042	0.014	0.099	0.029
	ρ	0.475	0.484	0.146	0.463	0.173	0.093	0.755	0.278	0.114	0.281	0.149	0.436	0.250
		(0.023)	(0.027)	(0.026)	(0.027)	(0.026)	(0.025)	(0.021)	(0.025)	(0.025)	(0.026)	(0.026)	(0.025)	(0.024)
C_t	ν	9.699	6.696	8.077	8.588	10.852	7.490	5.056	9.675	9.168	8.537	8.178	8.394	12.149
		(0.022)	(0.023)	(0.023)	(0.023)	(0.021)	(0.023)	(0.024)	(0.022)	(0.022)	(0.022)	(0.023)	(0.023)	(0.021)
	$\hat{L}_{cT}$	0.133	0.141	0.018	0.123	0.018	0.012	0.424	0.044	0.012	0.047	0.018	0.112	0.037

Notes: The entries (in the parentheses) are the QMLEs (and their standard deviations) obtained by the third-stage QML estimation: $\hat{\theta}_T := \text{argmax}_{\theta \in \Theta} L_{cT}(\theta|\hat{\boldsymbol{\gamma}}_T)$; see the Appendix for the estimation method, and $\hat{L}_{cT}$ denotes the fitted value of L_{cT}.

Table 11. The QMLEs of the dynamic C_N, C_G, C_G^s, and C_t.

		SP1 RS1	SP1 HS2	SP1 TW2	RS1 FT3	RS2 HS3	FT1 HS4	CA1 HS5
	κ_o	0.916	0.235	0.101	0.453	0.227	0.379	0.391
		(0.192)	(0.024)	(0.023)	(0.023)	(0.020)	(0.023)	(0.035)
C_N	κ_1	0.973	0.823	0.757	0.732	0.235	0.896	0.961
		(0.007)	(0.078)	(0.220)	(0.112)	(0.387)	(0.054)	(0.020)
	κ_2	0.018	0.041	0.028	0.051	0.061	0.028	0.016
		(0.007)	(0.014)	(0.018)	(0.018)	(0.029)	(0.011)	(0.007)
	$\hat{L}_{cT}$	0.471	0.027	0.006	0.107	0.027	0.072	0.066
	κ_o	0.862	0.196	0.097	0.416	0.203	0.328	0.330
		(0.007)	(0.008)	(0.007)	(0.008)	(0.009)	(0.008)	(0.011)
C_G	κ_1	0.976	0.818	0.739	0.682	0.537	0.881	0.974
		(0.001)	(0.020)	(0.041)	(0.034)	(0.088)	(0.014)	(0.003)
	κ_2	0.017	0.038	0.032	0.054	0.040	0.031	0.011
		(0.001)	(0.003)	(0.004)	(0.004)	(0.005)	(0.003)	(0.001)
	$\hat{L}_{cT}$	0.433	0.030	0.008	0.098	0.026	0.065	0.059
	κ_o	0.891	0.219	0.094	0.422	0.207	0.366	0.378
		(0.011)	(0.020)	(0.017)	(0.015)	(0.017)	(0.020)	(0.026)
C_G^s	κ_1	0.967	0.902	0.754	0.706	0.228	0.955	0.969
		(0.001)	(0.012)	(0.055)	(0.022)	(0.083)	(0.007)	(0.003)
	κ_2	0.022	0.025	0.019	0.056	0.069	0.016	0.016
		(0.001)	(0.003)	(0.003)	(0.003)	(0.005)	(0.001)	(0.001)
	$\hat{L}_{cT}$	0.475	0.028	0.007	0.104	0.029	0.075	0.070
	κ_o	0.929	0.232	0.101	0.457	0.228	0.378	0.409
		(0.062)	(0.036)	(0.029)	(0.030)	(0.028)	(0.035)	(0.069)
	κ_1	0.974	0.891	0.761	0.683	0.263	0.898	0.974
		(0.007)	(0.069)	(0.265)	(0.162)	(0.450)	(0.076)	(0.016)
C_t	κ_2	0.017	0.030	0.028	0.056	0.063	0.025	0.013
		(0.005)	(0.015)	(0.022)	(0.023)	(0.035)	(0.014)	(0.007)
	ν	7.329	8.788	8.798	10.400	10.613	8.167	8.367
		(0.025)	(0.022)	(0.022)	(0.022)	(0.021)	(0.023)	(0.023)
	$\hat{L}_{cT}$	0.487	0.033	0.012	0.112	0.032	0.079	0.073

Notes: The entries (in the parentheses) are the QMLEs (and their standard deviations) obtained by the third-stage QML estimation: $\hat{\theta}_T := \text{argmax}_{\theta \in \Theta} L_{cT}(\theta|\hat{\boldsymbol{\gamma}}_T)$; see the Appendix for the estimation method, and $\hat{L}_{cT}$ denotes the fitted value of L_{cT}. In the estimation, we set κ_o to be positive for considering the concordance structure.

Table 12. Tests for the null hypothesis: $C_o = C_N$.

	SP1 RS1	SP1 FT2	SP1 CA2	SP1 NK2	SP1 HS2	SP1 TW2	RS1 FT3	RS1 CA3	RS1 NK3	RS2 HS3
M_τ	0.769	0.590	1.021	−0.027	0.858	0.535	−0.189	−0.014	−1.594	−0.508
M_{LU}	1.732	5.324	7.007	9.351	3.550	8.430	2.831	3.125	9.334	6.927
$M_{L(0.1)}$	0.457	0.204	0.791	1.046	1.423	1.342	0.297	0.215	1.006	1.764
$M_{L(0.3)}$	1.114	−0.725	−0.915	1.095	−0.603	−0.115	0.990	0.301	1.590	−0.378
$M_{L(0.5)}$	0.251	−0.941	−1.220	0.158	−0.786	−0.315	0.783	0.779	1.367	1.000
$M_{U(0.5)}$	0.275	0.334	0.591	0.792	0.314	0.312	−0.293	−0.371	−1.408	−0.483
$M_{U(0.7)}$	0.299	0.181	0.906	−1.028	−0.164	−1.064	−0.080	0.845	−1.408	0.175
$M_{U(0.9)}$	0.019	1.983*	2.090*	0.860	−0.368	1.503	1.006	0.793	1.556	0.018

	RS1 TW3	FT1 CA1	FT1 NK4	FT1 HS4	FT1 TW4	CA1 NK5	CA1 HS5	CA1 TW5	NK1 HS1	NK1 TW1
M_τ	−0.995	0.976	−0.365	0.625	0.242	−0.195	0.806	0.567	−0.074	−0.686
M_{LU}	13.051*	7.108	8.804	5.487	10.656	10.268	4.888	4.214	6.296	3.231
$M_{L(0.1)}$	2.629*	0.328	0.521	0.788	1.813	1.570	0.732	1.222	1.005	0.835
$M_{L(0.3)}$	0.508	−0.161	1.421	−0.533	−0.825	0.850	−0.837	0.067	0.565	−0.110
$M_{L(0.5)}$	0.797	−0.196	−0.007	0.183	0.516	0.195	0.052	0.700	0.163	0.862
$M_{U(0.5)}$	−2.015*	0.869	−0.051	0.977	0.652	0.360	0.858	0.250	−0.085	−0.165
$M_{U(0.7)}$	−1.430	1.168	−1.034	0.281	−0.199	−0.406	0.060	0.142	−0.318	0.232
$M_{U(0.9)}$	0.930	2.171*	1.334	1.076	0.916	1.724	0.887	0.466	1.683	−0.064

Notes: The symbol * represents significance at the 5% level. The 95% critical value of $\chi^2(6)$ is 12.5916.

Table 13. Tests for the null hypothesis: $C_o = C_G$.

	SP1 RS1	SP1 FT2	SP1 CA2	SP1 NK2	SP1 HS2	SP1 TW2	RS1 FT3	RS1 CA3	RS1 NK3	RS2 HS3
M_τ	1.457	1.031	1.350	0.159	1.172	0.537	0.425	−1.354	0.333	−0.108
M_{LU}	22.493*	10.056	12.573	12.159	13.701*	7.375	12.850*	8.465	12.936*	13.598*
$M_{L(0.1)}$	3.120*	2.408*	2.841*	1.707	2.445*	1.663	2.315*	1.719	2.285*	2.661*
$M_{L(0.3)}$	3.131*	0.913	0.605	1.611	0.500	0.249	1.761	2.013*	2.514*	0.693
$M_{L(0.5)}$	0.953	−0.344	−0.710	0.379	−0.289	−0.220	1.153	1.401	1.319	1.369
$M_{U(0.5)}$	0.916	0.909	1.073	1.024	0.747	0.376	0.095	−1.216	0.262	−0.057
$M_{U(0.7)}$	−0.411	−0.440	0.161	−1.263	−0.353	−1.295	−0.039	−1.777	−0.802	−0.216
$M_{U(0.9)}$	−2.392*	−0.870	−0.805	−0.794	−2.447*	0.152	−2.135*	−0.434	−1.891	−2.236*

	RS1 TW3	FT1 CA1	FT1 NK4	FT1 HS4	FT1 TW4	CA1 NK5	CA1 HS5	CA1 TW5	NK1 HS1	NK1 TW1
M_τ	−0.914	1.203	0.001	1.070	0.410	0.119	1.230	0.719	0.255	−0.374
M_{LU}	12.751*	13.440*	14.613*	14.081*	10.451	13.735*	13.437*	6.759	13.430*	11.965
$M_{L(0.1)}$	2.857*	2.722*	1.863	2.455*	2.257*	2.770*	2.360*	1.857	2.836*	2.060*
$M_{L(0.3)}$	0.761	1.217	2.405*	0.944	−0.205	1.842	0.629	0.665	1.882	0.908
$M_{L(0.5)}$	0.763	0.081	0.420	0.745	0.693	0.562	0.621	0.866	0.566	1.171
$M_{U(0.5)}$	−1.943	1.158	0.386	1.512	0.867	0.751	1.389	0.484	0.326	0.202
$M_{U(0.7)}$	−1.645	0.241	−1.425	−0.111	−0.374	−0.886	−0.291	−0.155	−1.006	−0.197
$M_{U(0.9)}$	−0.406	−0.483	−1.016	−1.455	−0.367	−0.713	−1.630	−1.055	−1.170	−2.400*

Notes: The symbol * represents significance at the 5% level. The 95% critical value of $\chi^2(6)$ is 12.5916.

Table 14. Tests for the null hypothesis: $C_o = C_G^s$.

	SP1 RS1	SP1 FT2	SP1 CA2	SP1 NK2	SP1 HS2	SP1 TW2	RS1 FT3	RS1 CA3	RS1 NK3	RS2 HS3
M_τ	1.073	1.083	1.385	0.022	0.935	0.524	0.283	0.569	−1.249	−0.141
M_{LU}	11.393*	21.171*	19.449*	7.320	1.090	6.514	16.666*	22.097*	10.033	7.705
$M_{L(0.1)}$	−2.302*	−2.807*	−2.268*	−0.816	−0.341	−0.001	−2.797*	−2.871*	−0.681	−0.170
$M_{L(0.3)}$	0.059*	−1.231	−1.530	0.445	−0.547	−0.064	0.098	−0.437	0.928	−0.383
$M_{L(0.5)}$	0.459	−0.329	−0.715	0.229	−0.349	−0.056	1.236	1.148	1.402	0.890
$M_{U(0.5)}$	0.447	0.929	1.057	0.902	0.397	0.167	0.178	0.173	−1.102	−0.081
$M_{U(0.7)}$	1.702	1.766	2.264*	−0.440	0.622	−0.455	1.445	2.383*	−0.412	1.049
$M_{U(0.9)}$	2.127*	3.919*	3.758*	1.392	0.783	1.706	2.722*	2.613*	2.216*	1.125

	RS1 TW3	FT1 CA1	FT1 NK4	FT1 HS4	FT1 TW4	CA1 NK5	CA1 HS5	CA1 TW5	NK1 HS1	NK1 TW1
M_τ	−0.944	1.055	−0.172	0.755	0.261	0.003	0.953	0.601	0.262	−0.190
M_{LU}	8.751	23.330*	13.818*	13.386*	8.060	10.049	10.504	4.142	15.676*	10.357
$M_{L(0.1)}$	1.023	−2.830*	−2.100*	−1.959*	0.209	−1.028	−1.561	−0.604	−2.006*	−1.206
$M_{L(0.3)}$	0.038	−1.340	0.618	−1.204	−0.998	0.131	−1.351	−0.357	−0.160	−0.208
$M_{L(0.5)}$	0.697	−0.162	0.245	0.420	0.505	0.421	0.342	0.680	0.516	1.240
$M_{U(0.5)}$	−1.993*	0.920	0.230	1.058	0.731	0.656	0.987	0.356	0.322	0.414
$M_{U(0.7)}$	−0.909	2.230*	−0.010	1.281	0.204	0.586	1.090	0.607	1.022	1.449
$M_{U(0.9)}$	1.206	4.022*	2.401*	2.461*	1.302	2.757*	2.288*	1.005	3.247*	1.340

Notes: The symbol * represents significance at the 5% level. The 95% critical value of $\chi^2(6)$ is 12.5916.

Table 15. Tests for the null hypothesis: $C_o = C_t$.

	SP1 RS1	SP1 FT2	SP1 CA2	SP1 NK2	SP1 HS2	SP1 TW2	RS1 FT3	RS1 CA3	RS1 NK3	RS2 HS3
M_τ	0.265	0.407	0.655	−0.042	0.607	0.357	−0.133	−0.014	−0.931	−0.287
M_{LU}	1.637	1.828	1.940	6.350	1.232	4.905	1.074	2.291	4.356	4.569
$M_{L(0.1)}$	−0.233	−0.434	−0.178	−0.222	0.475	0.195	−0.506	−0.654	0.081	0.884
$M_{L(0.3)}$	0.847	−0.723	−0.995	0.755	−0.712	−0.367	0.668	0.075	1.309	−0.565
$M_{L(0.5)}$	0.075	−0.747	−1.017	0.104	−0.640	−0.289	0.565	0.607	1.178	0.877
$M_{U(0.5)}$	0.061	0.332	0.515	0.650	0.334	0.275	−0.282	−0.303	−1.157	−0.390
$M_{U(0.7)}$	0.011	0.067	0.417	−0.674	−0.153	−0.669	−0.157	0.319	−0.763	−0.014
$M_{U(0.9)}$	−0.150	0.272	0.236	−0.062	−0.204	0.056	0.045	−0.012	0.093	−0.133

	RS1 TW3	FT1 CA1	FT1 NK4	FT1 HS4	FT1 TW4	CA1 NK5	CA1 HS5	CA1 TW5	NK1 HS1	NK1 TW1
M_τ	−0.575	0.451	−0.276	0.400	0.142	−0.147	0.551	0.323	−0.102	−0.427
M_{LU}	7.053	1.016	6.045	3.297	6.430	2.286	3.683	1.851	1.101	2.909
$M_{L(0.1)}$	1.341	−0.783	−0.465	−0.196	0.733	0.519	−0.200	−0.050	0.039	0.122
$M_{L(0.3)}$	0.222	−0.586	1.099	−0.705	−1.100	0.569	−0.939	−0.300	0.262	−0.190
$M_{L(0.5)}$	0.745	−0.407	−0.065	0.149	0.446	0.142	0.084	0.567	0.062	0.833
$M_{U(0.5)}$	−1.635	0.441	−0.096	0.808	0.535	0.274	0.757	0.141	−0.143	−0.099
$M_{U(0.7)}$	−0.799	0.382	−0.661	0.040	−0.246	−0.345	−0.045	−0.121	−0.326	0.078
$M_{U(0.9)}$	−0.050	0.260	0.062	0.019	−0.030	0.118	−0.009	−0.122	0.144	−0.119

Notes: The 95% critical value of $\chi^2(6)$ is 12.5916.

interpreting the concordance structure of the return combinations captured by the positive and significant M_τ test statistics for the null of $C_o = C_I$. This also indicates that, as shown by the simulation and implied by the discussion of Section 3.1, the tail-dependence tests are more important than the concordance test in discriminating between various copula models.

Second, the M_{LU}, $M_L(u)$, and $M_U(u)$ tests accept the null of $C_o = C_N$ for sixteen out of the twenty return combinations; the only exceptions include SP**1**-FT**2**, SP**1**-CA**2**, RS**1**-TW**3**, and FT**1**-CA**1**. By contrast, the null of $C_o = C_G$ is accepted by all these tests for only three return combinations: SP**1**-NK**2**, SP**1**-TW**2**, and CA**1**-TW**5**. The null of $C_o = C_G^s$ is accepted by all these tests for seven return combinations: SP**1**-NK**2**, SP**1**-HS**2**, SP**1**-TW**2**, RS**2**-HS**3**, FT**1**-TW**4**, CA**1**-TW**5**, and NK**1**-TW**1**. Consequently, the normal copula evidently outperforms the Gumbel and Gumbel-survival copulae in this study. Given this good performance of C_N, it is not surprising to see that the t copula also outperforms the Gumbel and Gumbel-survival copulae because C_t is a generalization of C_N. Indeed, the tail-dependence tests are unable to reject the null of $C_o = C_t$ for all the return combinations considered.

Third, the above results enable us to characterize the cross-dependence structures of these return combinations (most of these combinations) by using the t copula (the normal copula). Because C_N and C_t are both of the U-shaped dependence, this implies that the co-movements of these stock markets will be further strengthened in turbulent periods. Moreover, this structure should symmetrically hold for both the downside and upside markets, and hence does not support the hypothesis of "correlation asymmetry".

Fourth, by sorting the Kendall's tau estimates, denoted as $\hat{\tau}$, of C_N for these twenty return combinations, we have the descending order: SP**1**-RS**1**($\hat{\tau}$=0.572), FT**1**-CA**1**(0.545), SP**1**-CA**2**(0.322), SP**1**-FT**2**(0.315), RS**1**-CA**3**(0.306), RS**1**-FT**3**(0.292), NK**1**-HS**1**(0.287), FT**1**-HS**4**(0.235), CA**1**-HS**5**(0.221), CA**1**-NK**5**(0.181), FT**1**-NK**4**(0.179), NK**1**-TW**1**(0.161), RS**2**-HS**3**(0.144), SP**1**-HS**2**(0.139), RS**1**-NK**3**(0.111), SP**1**-NK**2**(0.094), CA**1**-TW**5**(0.095), FT**1**-TW**4**(0.072), SP**1**-TW**2**(0.061), RS**1**-TW**3**(0.059). For the dynamic C_N, $\hat{\tau}$ is computed as the sample average of the estimates of the dynamic Kendall's tau: $\tau_t = \frac{2}{\pi}\arcsin(\rho_t)$. The t copula generates the same order and very similar Kendall's tau estimates. As noted in Section 3.1, the strength of the U-shaped dependence of C_N increases with Kendall's tau. This sort does not only show the variety of this strength across different return combinations, but also reveals the fact that this variety may be related to different factors, such as intra-national/international markets, overlapping/non-overlapping trading hours, and market scale.

Fifth, the significant $M_{L(0.1)}$ ($M_{U(0.9)}$) test statistics for the mis-specified Gumbel copula are all positive (negative). By contrast, the significant $M_{L(0.1)}$ ($M_{U(0.9)}$) test statistics for the mis-specified Gumbel-survival copula are all negative (positive). This illustrates that the Gumbel (Gumbel-survival) copula tends to under-estimate (over-estimate) the lower tail-dependence but to over-estimate (under-estimate) the upper tail-dependence for large $|u|$'s in this empirical study. Interestingly, this test result is consistent with the dissimilarity between the J-shape (L-shape) dependence implied by the Gumbel (Gumbel-survival) copula being rejected and the U-shaped dependence implied by the normal (or t) copula being accepted. This demonstrates that the $M_{L(u)}$ and $M_{U(u)}$ tests are useful in identifying the directions of copulae mis-specifications, as shown in the simulation.

Finally, we extend our analysis to the trivariate normal and t copulae based on the following return (marginal model) combinations: SP**1**-RS**1**-FT**2**, SP**1**-RS**1**-NK**2**, RS**1**-FT**3**-CA**3**, RS**1**-FT**3**-NK**3**, RS**1**-FT**3**-TW**3**, FT**1**-CA**1**-NK**5**, FT**1**-CA**1**-TW**4**, CA**1**-NK**5**-HS**5**, and CA**1**-NK**5**-TW**5**. These combinations are selected because the associated pairwise causality-in-mean and causality-in-variance test statistics are insignificant at the 5% level, so that their marginal models are directly applicable to the trivariate copula analysis.

The static trivariate C_N and C_t are, respectively, of the forms:

$$C_N(\boldsymbol{u};R) := \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{(2\pi)^{n/2}|R|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{v}^\top R^{-1}\boldsymbol{v}\right) \mathrm{d}v_n \ldots \mathrm{d}v_1$$

and

$$C_t(\boldsymbol{u};R,\nu) := \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{n/2}|R|^{1/2}} \left(1+\frac{1}{\nu}\boldsymbol{v}^\top R^{-1}\boldsymbol{v}\right)^{-\left(\frac{\nu+n}{2}\right)} \mathrm{d}v_n \ldots \mathrm{d}v_1,$$

where $n=3$, $\boldsymbol{u}=(u_1,u_2,u_3)^\top$, $\boldsymbol{v}=(v_1,v_2,v_3)^\top$, and R denotes the 3×3 correlation matrix with the (i,j)-th element $\rho_{ij}=1$ if $i=j=1,2,3$ and $\rho_{ij}=\rho_{ji}\in(0,1)$ if $i\neq j$. By using the test of Bera and Kim (2002), we observe that the ρ_{12}-CCC hypothesis is rejected for SP**1**-RS**1**-FT**2**, SP**1**-RS**1**-NK**2**, RS**1**-FT**3**-CA**3**, RS**1**-FT**3**-NK**3**, and RS**1**-FT**3**-TW**3**, the ρ_{13}-CCC hypothesis is rejected for CA**1**-NK**5**-HS**5**, and the ρ_{23}-CCC hypothesis is rejected for SP**1**-RS**1**-FT**2** at the 5% significance level. In these cases, we replace the static ρ_{ij} with the DCC coefficient $\rho_{ij,t}$, which takes the form (21). The QMLEs of the trivariate normal and t copulae are shown in Table 16. The causality and CCC test statistics are not reported for the sake of brevity.

Unlike the bivariate case, the multivariate Kendall's tau in (27) does not have a closed form for the trivariate normal and t copulae to the best of our knowledge, so that it is difficult to perform the multivariate M_τ test in this situation. Nonetheless, we may still check these two trivariate copulae by using the multivariate M_{LU}, $M_{L(u)}$, and $M_{U(u)}$ tests. The latter may be more important than the former in discriminating between two non-C_I copulae, as shown in the bivariate copula analysis. We show these tail-dependence test statistics in Table 17. For the trivariate normal copulae, this table indicates that the M_{LU} test statistic is insignificant at the 5% level for all but the case of RS**1**-FT**3**-NK**3** that has a marginally significant test statistic; the $M_{L(u)}$ and $M_{U(u)}$ test statistics are also insignificant in most cases except for the $M_{L(0.3)}$ test statistic which is significant for the combinations SP**1**-RS**1**-NK**2**, RS**1**-FT**3**-NK**3**, and FT**1**-CA**1**-NK**5**. In comparison, the M_{LU}, $M_{L(u)}$, and $M_{U(u)}$ test statistics for the trivariate t copulae are insignificant for all the cases considered. As such, the trivariate C_N and C_t perform quite well in this empirical example. Note that these two trivariate copulae also have the U-shaped dependence at the 45^o line: $u_1 = u_2 = u_3 = u \in (0, 1)$. Therefore, similar to the bivariate case, this extended analysis shows that the stock market relationships tend to be symmetrically strengthened in both the downside and upside markets.

6 CONCLUSION

In empirical finance, there is a rapidly growing interest in studying the cross-dependence structures of financial returns using copulae because of their modelling flexibility. To avoid the problems caused by copula mis-specification, it is important to check copulae by using formal statistical tests. In this paper, we propose a flexible class of moment-based copula tests in a generalized parametric copula-based multivariate dynamic context. The proposed method takes into consideration the estimation uncertainty, which is quite essential but has often been ignored in related studies. This method can be applied to generate various tests with distinctive power directions. On the basis of Kendall's tau, we obtain the concordance test that checks copulae in the direction of concordance (or dis-concordance). On the basis of the conditional probabilities of quantile-exceedances, we also develop a set of tail-dependence tests that detect copulae in their tail properties, either individually or simultaneously. These tests may be useful for risk management and other financial applications. The simulation shows the importance of correcting for the effect of estimation uncertainty in testing copulae and provides evidence that supports the validity of our tests. Finally, we also apply our tests to an empirical study on stock market relationships. This empirical study shows that the normal and t copulae outperform the Gumbel and Gumbel-survival copulae in characterizing the cross-dependence structures of stock index returns.

Table 16. The QMLEs of the trivariate normal and t copulae.

		SP**1** RS**1** FT**2**	SP**1** RS**1** NK**2**	RS**1** FT**3** CA**3**	RS**1** FT**3** NK**3**	RS**1** FT**3** TW**3**	FT**1** CA**1** NK**5**	FT**1** CA**1** TW**4**	CA**1** NK**5** HS**5**	CA**1** NK**5** TW**5**
C_N	θ_1	0.912 (0.032)	0.914 (0.032)	0.447 (0.019)	0.453 (0.021)	0.453 (0.021)	0.740 (0.006)	0.740 (0.006)	0.278 (0.018)	0.279 (0.018)
	θ_2	0.974 (0.005)	0.973 (0.005)	0.530 (0.264)	0.720 (0.116)	0.721 (0.118)	0.268 (0.018)	0.113 (0.020)	0.395 (0.036)	0.143 (0.022)
	θ_3	0.017 (0.003)	0.018 (0.003)	0.044 (0.018)	0.051 (0.017)	0.050 (0.017)	0.274 (0.018)	0.146 (0.019)	0.968 (0.021)	0.232 (0.022)
	θ_4	0.479 (0.017)	0.143 (0.027)	0.459 (0.019)	0.170 (0.022)	0.093 (0.021)	.	.	0.012 (0.007)	.
	θ_5	0.435 (0.019)	0.164 (0.023)	0.726 (0.014)	0.225 (0.035)	0.090 (0.023)	.	.	0.410 (0.015)	.
	θ_6	0.286 (0.384)	.	.	.	.	.	.	.	.
	θ_7	0.042 (0.019)	.	.	.	.	.	.	.	.
	$\hat{L}_{cT}$	0.608	0.485	0.505	0.136	0.113	0.438	0.403	0.170	0.071
C_t	θ_1	0.913 (0.059)	0.922 (0.073)	0.448 (0.043)	0.454 (0.051)	0.450 (0.050)	0.757 (0.021)	0.756 (0.017)	0.282 (0.026)	0.281 (0.030)
	θ_2	0.974 (0.026)	0.974 (0.013)	0.443 (0.863)	0.663 (0.436)	0.674 (0.236)	0.273 (0.033)	0.120 (0.028)	0.594 (0.328)	0.147 (0.026)
	θ_3	0.016 (0.014)	0.017 (0.007)	0.046 (0.054)	0.054 (0.077)	0.054 (0.038)	0.275 (0.033)	0.145 (0.028)	0.991 (0.005)	0.224 (0.028)
	θ_4	0.473 (0.043)	0.149 (0.033)	0.460 (0.034)	0.174 (0.031)	0.093 (0.027)	.	.	0.007 (0.003)	.
	θ_5	0.440 (0.049)	0.167 (0.030)	0.741 (0.021)	0.230 (0.037)	0.089 (0.028)	.	.	0.412 (0.026)	.
	θ_6	0.310 (0.405)	.	.	.	.	.	.	.	.
	θ_7	0.046 (0.024)	.	.	.	.	.	.	.	.
	ν	8.880 (0.021)	9.411 (0.021)	7.241 (0.021)	11.153 (0.020)	10.784 (0.020)	7.725 (0.021)	7.805 (0.020)	14.093 (0.019)	9.771 (0.020)
	$\hat{L}_{cT}$	0.631	0.506	0.539	0.147	0.125	0.471	0.438	0.188	0.086

Notes: The entries (in the parentheses) are the QMLEs (and their standard deviations) obtained by the third-stage QML estimation: $\hat{\theta}_T := \operatorname{argmax}_{\theta\in\Theta} L_{cT}(\theta|\hat{\boldsymbol{\gamma}}_T)$; see the Appendix for the estimation method. The DCC specification is of the form: $\rho_{ij,t} = (1-\kappa_{ij,1}-\kappa_{ij,2})\kappa_{ij,o} + \kappa_{ij,1}\rho_{ij,t-1} + \kappa_{ij,2}\frac{\sum_{k=1}^{m}\varepsilon_{i,t-k}\varepsilon_{j,t-k}}{\sqrt{\left(\sum_{k=1}^{m}\varepsilon_{i,t-k}^2\right)\left(\sum_{k=1}^{m}\varepsilon_{j,t-k}^2\right)}}$, where $ij = 12, 13$, or 23, and $m = 2$. Let ρ_{12}, ρ_{13}, and ρ_{23} be the static correlation coefficients. For SP**1**-RS**1**-FT**2**, $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7) = (\kappa_{12,o},\kappa_{12,1},\kappa_{12,2},\rho_{13},\kappa_{23,o},\kappa_{23,1},\kappa_{23,2})$. For SP**1**-RS**1**-NK**2**, RS**1**-FT**3**-CA**3**, RS**1**-FT**3**-NK**3**, RS**1**-FT**3**-TW**3**, $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) = (\kappa_{12,o},\kappa_{12,1},\kappa_{12,2},\rho_{13},\rho_{23})$. For FT**1**-CA**1**-NK**5**, FT**1**-CA**1**-TW**4**, CA**1**-NK**5**-TW**5**, $(\theta_1,\theta_2,\theta_3) = (\rho_{12},\rho_{13},\rho_{23})$. For CA**1**-NK**5**-HS**5**, $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) = (\rho_{12,o},\kappa_{13,o},\kappa_{13,1},\kappa_{13,2},\rho_{23})$. $\hat{L}_{cT}$ denotes the fitted value of L_{cT}.

Table 17. Tests for the trivariate C_N and C_t.

		SP**1** RS**1** FT**2**	SP**1** RS**1** NK**2**	RS**1** FT**3** CA**3**	RS**1** FT**3** NK**3**	RS**1** FT**3** TW**3**	FT**1** CA**1** NK**5**	FT**1** CA**1** TW**4**	CA**1** NK**5** HS**5**	CA**1** NK**5** TW**5**
C_N	M_{LU}	4.445	7.646	6.327	12.778*	8.618	12.392	6.603	9.650	2.131
	$M_{L(0.1)}$	−0.030	0.671	1.076	0.617	1.800	1.859	0.997	1.189	1.030
	$M_{L(0.3)}$	1.068	1.981*	1.217	2.288*	0.519	2.292*	−0.824	1.170	0.680
	$M_{L(0.5)}$	0.082	1.148	0.843	1.834	0.672	0.613	1.180	0.239	0.888
	$M_{U(0.5)}$	0.104	−0.333	−0.075	−1.071	−0.662	0.258	0.065	0.404	−0.280
	$M_{U(0.7)}$	0.231	−0.582	0.787	−0.971	−0.825	0.239	−0.185	−0.840	−0.069
	$M_{U(0.9)}$	1.456	0.859	1.658	1.124	1.001	1.587	0.510	1.146	0.284
C_t	M_{LU}	2.669	4.419	0.801	2.678	2.226	3.445	6.495	4.948	0.998
	$M_{L(0.1)}$	−0.783	−0.421	−0.212	−0.307	0.807	0.497	−0.403	0.407	0.011
	$M_{L(0.3)}$	0.495	1.575	0.422	1.143	0.266	1.379	−1.264	0.809	0.410
	$M_{L(0.5)}$	−0.002	0.919	0.367	1.008	0.705	0.302	0.775	0.179	0.899
	$M_{U(0.5)}$	0.019	−0.270	−0.132	−0.606	−0.407	0.040	−0.081	0.234	−0.171
	$M_{U(0.7)}$	0.003	−0.255	0.110	−0.330	−0.256	−0.048	−0.174	−0.268	−0.072
	$M_{U(0.9)}$	0.082	0.013	0.073	0.037	0.028	0.067	−0.037	0.052	−0.026

Notes: The symbol * represents significance at the 5% level. The 95% critical value of $\chi^2(6)$ is 12.5916.

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APPENDIX: A THREE-STAGE ESTIMATION METHOD

This method first estimates α_{io} by maximizing the Gaussian quasi-likelihood function $L_{\alpha_i T}(\alpha_i) := -\frac{1}{2}\ln 2\pi - \frac{1}{2T}\sum_{t=1}^{T}\ln h_{it} - \frac{1}{2T}\sum_{t=1}^{T} h_{it}^{-1}(y_{it}-m_{it})^2$, then estimates β_{io} by maximizing the $\hat{\alpha}_{iT}$-based quasi-likelihood function $L_{\beta_i T}(\beta_i|\hat{\alpha}_{iT}) := \frac{1}{T}\sum_{t=1}^{T}\ln f_{\varepsilon_i}(\hat{\varepsilon}_{it}|\boldsymbol{x}_t;\beta_i)$, and finally estimates θ_o by maximizing the $\hat{\boldsymbol{\gamma}}_T$-based quasi-likelihood function $L_{cT}(\theta|\hat{\boldsymbol{\gamma}}_T) := \frac{1}{T}\sum_{t=1}^{T}\ln c(\hat{\boldsymbol{u}}_t|\boldsymbol{x}_t;\theta)$, where $c(\boldsymbol{u}|\boldsymbol{x}_t;\theta) := \frac{\partial^n}{\partial u_1 \partial u_2 \ldots \partial u_n} C(\boldsymbol{u}|\boldsymbol{x}_t;\theta)$. It is standard to argue the $\sqrt{T}$-consistency of these QMLEs by the regularity conditions of uniform convergence and identifiable uniqueness; see, e.g., White (1994). The following discussion is based on assumption [A] and the null hypothesis. We also denote $\psi^o_{\alpha,it} = \psi_{\alpha,it}|_{\alpha_i=\alpha_{io}}$, $\psi^o_{\beta,it} = \psi_{\beta,it}|_{\gamma_i=\gamma_{io}}$, and $\psi^o_{\theta t} = \psi_{\theta t}|_{\lambda=\lambda_o}$.

By expanding the estimating equation of $\hat{\alpha}_{iT}$: $\frac{1}{T}\sum_{t=1}^{T}\{\hat{w}_{it}\hat{\varepsilon}_{it} + \frac{1}{2}\hat{z}_{it}(\hat{\varepsilon}^2_{it}-1)\} = 0$ around $\alpha_i = \alpha_{io}$, it is standard to show that

$$\psi_{\alpha,it} = \left\{\mathbb{E}[w_{it}w_{it}^\top] + \frac{1}{2}\mathbb{E}[z_{it}z_{it}^\top]\right\}^{-1}\left\{w_{it}\varepsilon_{it} + \frac{1}{2}z_{it}(\varepsilon^2_{it}-1)\right\}; (A1)$$

see, e.g., Bollerslev and Wooldridge (1992). We can also expand the estimating equation of $\hat{\beta}_{iT}$: $\frac{1}{T}\sum_{t=1}^{T}\nabla_{\beta_i}\ln f_{\varepsilon_i}(\hat{\varepsilon}_{it}|\boldsymbol{x}_t;\hat{\beta}_{iT}) = 0$ around $\gamma_i = \gamma_{io}$, utilize (9) and $(A1)$, and apply the information matrix equality: $\mathbb{E}[\nabla_{\beta_i^\top} l_{\beta,it}] = -\mathbb{E}[l_{\beta,it}l_{\beta,it}^\top]$ as $\gamma_i = \gamma_{io}$ to show that

$$\psi_{\beta,it} = \mathbb{E}[l_{\beta,it}l_{\beta,it}^\top]^{-1}\left\{l_{\beta,it} + \zeta^o_{it}\psi_{\alpha,it}\right\}, (A2)$$

where $l_{\beta,it} := \nabla_{\beta_i}\ln f_{\varepsilon_i}(\varepsilon_{it}|\boldsymbol{x}_t;\beta_i)$ and $\zeta^o_{it} := -\left\{\mathbb{E}\left[\left(\frac{\partial l_{\beta,it}}{\partial \varepsilon_{it}}\right)w_{it}^\top\right] + \frac{1}{2}\mathbb{E}\left[\left(\frac{\partial l_{\beta,it}}{\partial \varepsilon_{it}}\right)z_{it}^\top\right]\right\}\Big|_{\gamma_i=\gamma_{io}}$. Similarly, we can expand the estimating equation of $\hat{\theta}_T$: $\frac{1}{T}\sum_{t=1}^{T}\nabla_\theta \ln c(\hat{\boldsymbol{u}}_t|\boldsymbol{x}_t;\hat{\theta}_T) = 0$ around $\lambda = \lambda_o$, utilize (9), (10), $(A1)$, and $(A2)$, and apply the information matrix equality: $\mathbb{E}[\nabla_{\theta^\top} l_{\theta t}] = -\mathbb{E}[l_{\theta t}l_{\theta t}^\top]$ as $\lambda = \lambda_o$ to obtain the result:

$$\psi_{\theta t} = \mathbb{E}[l_{\theta t}l_{\theta t}^\top]^{-1}\left\{l_{\theta t} + \sum_{i=1}^{n}\left(\xi^o_{\alpha i}\psi_{\alpha,it} + \xi^o_{\beta i}\psi_{\beta,it}\right)\right\}, (A3)$$

where $l_{\theta t} := \nabla_\theta \ln c(\boldsymbol{u}_t|\boldsymbol{x}_t;\theta)$, $\xi^o_{\alpha i} := \mathbb{E}\left[\left(\frac{\partial l_{\theta t}}{\partial u_{it}}\right)(\nabla_{\alpha_i^\top}u_{it})\right]$, and $\xi^o_{\beta i} := \mathbb{E}\left[\left(\frac{\partial l_{\theta t}}{\partial u_{it}}\right)(\nabla_{\beta_i^\top}u_{it})\right]$ are evaluated at $\lambda = \lambda_o$.

Note that, given (9), (10), and (11), we can further apply the martingale-difference central limit theorem and the Cramér-Wold device to show $\sqrt{T}(\hat{\alpha}_{iT} - \alpha_{io}) \xrightarrow{d} N(0, \Sigma^o_{\alpha i})$, $\sqrt{T}(\hat{\beta}_{iT} - \beta_{io}) \xrightarrow{d} N(0, \Sigma^o_{\beta i})$, and $\sqrt{T}(\hat{\theta}_T - \theta_o) \xrightarrow{d} N(0, \Sigma^o_\theta)$ in which the asymptotic variance-covariance matrices $\Sigma^o_{\alpha i} := \mathbb{E}[\psi^o_{\alpha,it}\psi^{o\top}_{\alpha,it}]$, $\Sigma^o_{\beta i} := \mathbb{E}[\psi^o_{\beta,it}\psi^{o\top}_{\beta,it}]$, and $\Sigma^o_\theta := \mathbb{E}[\psi^o_{\theta t}\psi^{o\top}_{\theta t}]$ can be consistently estimated by the simple outer-product estimators: $\hat{\Sigma}_{\alpha i} := T^{-1}\sum_{t=1}^{T}\hat{\psi}_{\alpha,it}\hat{\psi}_{\alpha,it}^\top$, $\hat{\Sigma}_{\beta i} := T^{-1}\sum_{t=1}^{T}\hat{\psi}_{\beta,it}\hat{\psi}_{\beta,it}^\top$, and $\hat{\Sigma}_{\theta T} := T^{-1}\sum_{t=1}^{T}\hat{\psi}_{\theta t}\hat{\psi}_{\theta t}^\top$, respectively. Accordingly, we can easily evaluate the statistical significance of $\hat{\alpha}_{iT}$, $\hat{\beta}_{iT}$, and $\hat{\theta}_T$. □

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